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ON INTERSECTION INVARIANTS OF A COMPLEX AND ITS COMPLEMENTARY SPACE

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The intersection theory set up by Lefschetz[1] has made possible the introduction of a new topological invariant—the *ring of cycles of a manifold*.[2] The product of two elements of the ring is their intersection. The sum of two elements is the ordinary sum of two elements of a Betti group, if the two elements are of the same dimensionality; otherwise the sum is defined quite formally. We shall prove that: *if two manifolds are imbedded in a Euclidean (or spherical) space and the rings of these manifolds are different (not isomorphic), the corresponding complementary spaces are not homeomorphic.* To do this we construct in the complementary space, provided its dimensionality is sufficiently high (at least $2n + 2$, if n is the dimensionality of the manifold), an invariant ring and show that this ring is isomorphic to the ring of the manifold. The ring constructed in the complementary space enables us to define *the ring of intersections* not only for manifolds *but also for complexes.* Our discussion will be confined to the field of coefficients modulo 2. The extension to other rings of coefficients—the ring of all integers or the rings modulo m—requires only unimportant modifications in formulation and argument. We consider a closed manifold M^n, an imbedding r-dimensional Euclidean space (or an r-dimensional sphere) R^r, and the complementary space $R^r - M^n = Q^r$. In §§1 and 3 we assume that Q^r is the space complementary to a certain closed n-dimensional set F^n rather than to the manifold M^n.

1. The result of the usual intersection of two cycles of the complementary space is always a bounding cycle. Indeed let X and Y be two cycles of the complementary space: $X \subset Q$, $Y \subset Q$. Let A be a chain in R^r bounded by X: $A \to X$ on R^r. As the intersection of X and Y in Q^r does not differ from that in R^r we obtain by applying to AY the boundary relation for intersections:[3] $AY \to XY$ on R^r. But AY lies entirely in Q, as $Y \subset Q$. Therefore $AY \to XY$ on Q, i.e. $XY \sim 0$ in the complementary space. We define now the multiplication for cycles of the complementary space Q. Let A^{k+1} and B^{l+1} be chains

[1] S. Lefschetz, "Intersections and transformations of complexes and manifolds," *Trans. Amer. Math. Soc.* 28 (1926). See also Lefschetz, *Topology*, Amer. Math. Soc. Colloq. Publ. (1930).

[2] Hopf, H. "Zur Algebra der Abbildungen von Mannigfaltigkeiten," *Journ. reine angew. Math.*, 163 (1930), pp. 71–88.

[3] Lefschetz, "Intersections . . ." loc. cit. p. 12.

in R^r bounded by X^k and Y^l respectively. The intersection of these chains will be a chain AB. Let us call the product of X and Y in Q a cycle $Z = [XY] = (AB)^\bullet = XB + AY$, where by $(AB)^\bullet$ is meant the boundary of AB. This cycle lies entirely in Q, since $X \subset Q$, $Y \subset Q$. The dimensionality of the cycle Z is $k + l + 1 - r$.

In this way the definite cycle $[XY] = Z^{k+l+1-r}$ corresponds to a pair X^k, Y^l of cycles of Q.

THEOREM. *The class of homologies defined by the product* $[XY]$ *is independent of the particular chains* A *and* B.

In other words if A' and B' are other chains bounded by X and Y respectively and Z' is defined by means of A' and B' then $Z \sim Z'$ on Q.

PROOF. It is sufficient to assume that A and A' are different and that B' coincides with B. Let $A \to X$ on R, $A' \to X$ on R. $A - A'$ is a cycle in R so that $\pi \to A - A'$ on R where π is a chain in R. The formula for the boundary of an intersection then gives

$$((A - A')\cdot B)^\bullet = (A - A')\cdot B^\bullet = (A - A')Y = (\pi Y)^\bullet \text{ on } R$$

or

$$((A - A')\cdot B)^\bullet = Z - Z' = (\pi Y)^\bullet.$$

This means that $Z \sim Z'$ on Q, provided πY lies entirely in Q.

THEOREM. *If at least one of cycles* X, Y *bounds on* Q *so does their product.*

PROOF. Let $X \sim 0$ on Q. Then there exists a chain A lying entirely in Q such that $A \to X$ on Q. Let us now use this A in order to obtain $[XY]$. By definition $[XY] = (AB)^\bullet$. But $A \subset Q$. Therefore $AB \subset Q$ and $Z = [XY] \sim 0$ on Q.

COROLLARY. If $X \sim X'$ on Q, then $[XY] \sim [X'Y]$ on Q. For $[(X - X')Y] \sim 0$ on Q, and the intersection operation is distributive.

The last corollary enables us to consider as the multiplied elements not *cycles*, but *classes of homologous cycles*. We thus obtain the ring of homologies in the complementary space Q. However, the ring of Q as so far defined is not a topological invariant of Q alone. This is so because the chains A and B used to define the product $[XY]$ need not be contained in Q but may be anywhere in R. In order to obtain a ring of Q which is an invariant of Q alone it is necessary to define the ring of Q *intrinsically*, i.e. in terms of chains entirely contained in Q. This will be done in §3 by means of infinite chains and cycles. The dimensionality of Q will be subjected there to the condition mentioned above.

2. Let us now prove that if Q^r is the space complementary in R^r to a manifold M^n the ring constructed in Q^r is isomorphic to the ring of the manifold.

Let us consider a fundamental set of k-dimensional cycles for the manifold M^n (i.e. a base for homologies of dimensionality k) $x_1^k, x_2^k, \cdots, x_{p^k}^k$. According to the Poincaré-Veblen theorem if $\bar{x}_1^{n-k}, \bar{x}_2^{n-k}, \cdots, \bar{x}_{p^k}^{n-k}$ is the fundamental set of $(n - k)$-dimensional cycles of M^n and a_{ij} is the intersection-

number $\chi(x_i^k, \bar{x}_j^{n-k})$ of cycles x_i and $\bar{x}_j$, then the determinant $|a_{ij}| = 1$. In the same way we obtain according to the Alexander duality relation: if $X_1^{r-n+k-1}, \cdots, X_{p^k}^{r-n+k-1}$ is a fundamental set of $(r - n + k - 1)$-dimensional cycles of the complementary space Q, and V_{ij} is the looping coefficient of the cycles $\bar{x}_i^{n-k}$ and $X_j^{r-n+k-1}$, then the determinant $|V_{ij}| = 1$. Making use of this we can easily show that if we have the base $x_1^k, x_2^k, \cdots, x_{p^k}^k$ of the k-dimensional Betti group of M^n it is possible to choose in a unique way the base $\bar{x}_1^{n-k}, \bar{x}_2^{n-k}, \cdots, \bar{x}_{p^k}^{n-k}$ so that $a_{ij} = \delta_{ij}$, where δ_{ij} is the Kronecker symbol. We can choose similarly in a unique way the base $X_1^{n-r+k-1}, X_2^{r-n+k-1}, \cdots, X_{p^k}^{r-n+k-1}$ so that $V_{ij} = \delta_{ij}$. We suppose that our bases are thus chosen.

Let us establish now the one-to-one correspondence between the elements of the k-dimensional Betti group of the manifold and the elements of the $(r - n + k - 1)$-dimensional Betti group of the complementary space. For that purpose we relate to the base element x_i^k the base element $X_i^{r-n+k-1}$ and to the element $t^i x_i$ the homology class $X = t^i X_i$.

The correspondence just established is invariant; it does not depend on the base chosen for the k-dimensional Betti group. Let us examine the geometric nature of the cycle x^k which corresponds in this way to the cycle $X^{r-n+k-1} \subset Q$. We assume that M^n lies in R^r polyhedrally, i.e. every simplex of M^n is a Euclidean simplex in R^r.[4] Let A^{r-n+k} be a polyhedral complex in R^r bounded by $X^{r-n+k-1}$ and in a general position with respect to the manifold M^n. The intersection of the chain A with M defines a cycle $^*x = MA$ on M^n. The dimensionality of $^*x$ is $r - n + k + n - r = k$.

THEOREM. *Let x^k be the cycle corresponding in the above way to $X^{r-n+k-1}$. Then $^*x^k \sim x^k$ on M^n.*

From this theorem it follows on the one hand that the class of homologies defined by x^k does not depend on the special choice of the fundamental set of M^n and on the other that $^*x^k$ does not depend, as a class of homologies, on the chain A bounded by $X^{r-n+k-1}$.

PROOF. It is sufficient to prove the theorem for the cycles corresponding to the cycles of the fundamental set $X_1, X_2, \cdots, X_{p^k}$. Let us suppose that $X = X_l$ and $^*x \sim t^i x_i$. Since $V(\bar{x}_i^{n-k}, X_j^{r-n+k-1}) = V_{ij} = \delta_{ij}$ the looping coefficient $V(\bar{x}_j, X) = \delta_{jl}$. $V(\bar{x}_j, X)$ is the number mod 2 of the points of intersection of the cycle $\bar{x}_j$ with the chain bounded by X, for example with A. But it is also the number (mod 2) of points of intersection of $\bar{x}_j$ with the cycle $^*x$ on M. i.e. $\chi(^*x, \bar{x}_j)$. Therefore $\chi(^*x, \bar{x}_j) = \delta_{jl}$. Since $\chi(x_i, \bar{x}_j) = \delta_{ij}$ we obtain

$$\chi(^*x, \bar{x}_j) = \chi(t^i x_i, \bar{x}_j) = t^i \delta_{ij} = t^j,$$

or $t^j = \delta_{jl}$. This means that $t^l = 1$, $t^i = 0$ $(i \neq l)$ whence $^*x \sim x$ on M.

It follows immediately that the ring of the complementary space Q defined in §1 is isomorphic to the ring of M. Let X^k, Y^l be two cycles of Q. Their

[4] This assumption is not essential and may easily be dispensed with.

product is $[X^k, Y^l] = Z^{k+l+1-r}$. Let $x^{n+k+1-r}$ be the cycle on M corresponding (in the sense stated) to X and $y^{n+l+1-r}$ the cycle corresponding to Y. We shall prove that the cycle Z corresponds to xy, i.e. the product of two cycles in Q corresponds to the intersection of the corresponding cycles on M. This will complete the proof of the theorem.

Let A be a chain in R bounded by X, B a chain bounded by Y. We have shown that $x \sim MA$, $y \sim MB$. In order to obtain z we must take the intersection of M with a chain bounded by Z. But $Z = (AB)^{\bullet}$. Hence we can take AB as such a chain. Thus $z = M \cdot AB$. The intersection of M with AB in R is geometrically the same set of points (chain) as the intersection of MA with MB on M. Therefore

$$\begin{aligned} z &= M \cdot AB \text{ on } R \\ &= MA \cdot MB \text{ on } M \\ &= x \cdot y, \end{aligned}$$

which proves the theorem.

3. In this paragraph we give the intrinsic definition of the ring of Q, without making use of chains lying in R^r (such as A and B). Q^r is the open (relative) manifold, not compact, complementary to the closed set F^n (the set F^n lying entirely inside the Euclidean r-space R^r). In order to construct the invariant ring in Q we shall use the notion of an infinite chain.[5]

By an *infinite chain of the complementary space* Q is meant a chain consisting of an enumerable number of simplexes of Q and such that nearly all its simplexes (i.e. all with the exception of a finite number) lie wholly outside every compact part of Q. Let the simplexes of the infinite chain be enumerated. According to our definition all simplexes with sufficiently large indices lie wholly outside any given compact part of Q. The *boundary* of an infinite chain is defined as the sum of the boundaries of the separate simplexes. An *infinite cycle* is an infinite chain whose boundary vanishes.

We define now the ring of the complementary space Q as follows. Let X^k and Y^l be two finite cycles in Q. Let $\mathfrak{A}^{k+1}$ be an infinite chain in Q bounded by X^k (the existence of such a chain will be shown later) and $\mathfrak{B}^{k+1}$ an infinite chain in Q bounded by Y^l. We define the product of X^k and Y^l by the formula

$$[X^k Y^l] = \mathfrak{A}^{k+1} Y + X^k \mathfrak{B}^{l+1}.$$

We want to prove now that if $r \geqq 2n + 2$ the homology class of the product so defined does not depend on the particular chains $\mathfrak{A}$ and $\mathfrak{B}$. Let $'\mathfrak{A}^{k+1}$ be any other infinite chain bounded by X^k. Evidently it is sufficient to show that the product defined by means of $\mathfrak{A}$ and $\mathfrak{B}$ is homologous in the complementary space to the product defined by means of $'\mathfrak{A}$ and $\mathfrak{B}$, i.e. $\mathfrak{A}Y \sim {'\mathfrak{A}}Y$ on Q or

[5] See Lefschetz, *Topology*, Chapter VII.

$(\mathfrak{A} - '\mathfrak{A}) \cdot Y \sim 0$ on Q. $\mathfrak{A} - '\mathfrak{A}$ is an infinite cycle of dimensionality $k + 1$. We denote it by $\mathfrak{Z}$ and shall prove that if $r \geqq 2n + 2$ the intersection $\mathfrak{Z}^{k+1} \cdot Y \sim 0$ on Q.

We observe that since X^k is not homologous to zero its dimensionality $k \geqq r - n - 1 \geqq 2n + 2 - n - 1 = n + 1$. Let all the simplexes of $\mathfrak{Z}$ be enumerated and let $\mathfrak{Z}$ be divided into two parts $'\mathfrak{Z}_N$ and $''\mathfrak{Z}_N$: $\mathfrak{Z} = '\mathfrak{Z}_N + ''\mathfrak{Z}_N$; where $'\mathfrak{Z}_N$ is the finite chain containing all the simplexes of $\mathfrak{Z}$ with the indices from 1 to N. Then we can find for every δ an N such that $''\mathfrak{Z}_N$ lies entirely within the δ-neighborhood U_δ of F.

We consider now the finite cycle $\zeta_N^k = ('\mathfrak{Z}_N^{k+1})^\bullet$. Since $\mathfrak{Z}^\bullet = 0$, $\zeta_N^k = (''\mathfrak{Z}_N^{k+1})^\bullet$, i.e. ζ_N^k lies entirely inside U_δ. It is known that for every ϵ we can find a δ such that every cycle ζ of dimensionality $\geqq n + 1$ lying in the δ-neighborhood of the n-dimensional closed set F is homologous to zero in the ϵ-neighborhood of F. (In order to prove this we may take the nerve of a suitable covering of F by closed sets geometrically realized near F and shift ζ on this nerve.) Therefore, since $k \geqq n + 1$, $\zeta_N^k \sim 0$ on U_ϵ, i.e. there exists a finite chain P^{k+1} in U_ϵ such that $P_N^{k+1} \rightarrow \zeta_N^k$ on U_ϵ. $'\mathfrak{Z}_N^{k+1} + P_N^{k+1}$ is a finite cycle in R^r. Let π^{k+2} be a chain in R^r bounded by this cycle: $\pi^{k+2} \rightarrow '\mathfrak{Z}_N^{k+1} + P_N^{k+1}$ on R^r and let ϵ be chosen small enough so that Y has no common points with U_ϵ. Then $\mathfrak{Z}^{k+1} \cdot Y = '\mathfrak{Z}_N^{k+1} \cdot Y$. But $(\pi Y)^\bullet = \pi^\bullet Y = '\mathfrak{Z}_N Y + P_N Y = '\mathfrak{Z}_N Y = \mathfrak{Z} Y$, since $P_N \subset U_\epsilon$ and has no common points with Y. Therefore $\mathfrak{Z}^{k+1} Y \sim 0$ on Q. The theorem is proved.

The ring defined in this way does not differ from the ring defined in §1 by means of chains A and B situated in R^r. In fact let A^{k+1} be a chain in R^r bounded by X^k. Consider the part of A^{k+1} which consists of all points of A^{k+1} that do not belong to F. We can triangulate this part in such a way that it will represent an infinite chain $\mathfrak{A}^{k+1}$ consisting of an enumerable number of simplexes, the boundary of which is the cycle X (perhaps subdivided).[6] This proves also the existence of an infinite chain of the complementary space Q bounded by any given cycle X^k of Q.

The intersection $\mathfrak{A}^{k+1} \cdot Y$ does not differ from the intersection $A^{k+1} \cdot Y$. Therefore the two definitions are equivalent.

Remark 1. The rules for the intersections of finite chains hold also for the intersections of finite chains with infinite chains as considered in this paragraph. In fact we have considered here only the intersection of finite parts of infinite chains.

Remark 2. If Q is the space complementary to the manifold M^n the ring of Q constructed by means of infinite chains $\mathfrak{A}$ and $\mathfrak{B}$ is isomorphic to the ring of M as before.

Remark 3. If F is a connected complex, the ring of Q may be defined invariantly even when $r = 2n + 1$. In fact as we have seen the product $[X^k Y^l]$ fails to be defined uniquely only if there exists in Q an infinite cycle $\mathfrak{Z}$ of di-

[6] See Alexandroff-Hopf, *Topologie*, Erster Band, pp. 143–147.

mensionality $k + 1$ whose intersection $\mathfrak{Z}\cdot Y$ does not bound in Q (or if for Y the same thing occurs for the corresponding infinite cycle of dimensionality $l + 1$). This is possible only when $k \leqq n$. On the other hand when $r = 2n + 1$, $k \geqq n$, since X^k does not bound in Q. Hence $k = n$. If $p < n$ the product $[X^n Y^{n+p}] \sim 0$ on Q (since its dimensionality $= p < n$). As for the product $[X^n Y^{2n}]$ we define it by the formula $[X^n Y^{2n}] = X^n$.[7] This is confirmed by the fact that for a connected complex there exists in Q only one cycle Y (more exactly only one class of homologies) of dimensionality $2n$ that is not homologous to zero.

If F is a connected manifold M the ring obtained in Q is isomorphic again to the ring of M. In fact in this case the cycle Y^{2n} corresponds in the way previously described to the manifold M itself.

4. The definition of the ring given in §1 enables us to introduce *the ring of a complex*. Let us consider a complex K^n. In order to obtain the ring of K^n we imbed it in a Euclidean space (sphere) $R^r (r \geqq 2n + 2)$ and take as the ring of the complex the ring of the complementary space Q^r defined as in §1 (by means of chains A and B situated in R). Taking account of the known theorem about the isotopy of n-dimensional homeomorphic sets in $(2n + 2)$-dimensional Euclidean space[8] we see easily that the ring thus defined is the same however we immerse K in R^r, i.e. is defined uniquely by R^r.

We want to prove now that this ring is independent of the dimensionality of the imbedding space R^r. It is sufficient to show that the two rings which are obtained when we imbed the complex in R^r and R^{r+1} are isomorphic. Let the ring of Q be defined by the intersections of base cycles

$$(1) \qquad X_1^k, X_2^k, \cdots, X_{pk}^k \ (r - n - 1 \leqq k \leqq r - 1)$$

of Betti groups. Let us now imbed R^r in R^{r+1}. We shall then obtain the space Q^{r+1} complementary to the complex. In order to get the base cycles of Q^{r+1} we proceed as follows.[9] Let X^k be any cycle of (1) and A^{k+1} a chain of R^r bounded by X^k. We construct now in R^{r+1} on both sides of the linear space R^r two cylinders, having as bases A^{k+1}, and whose generators are perpendicular to R^r and have the length h. The two cylinders together form a cylinder $'A^{k+2}$ whose height $= 2h$. We denote the boundary of $'A^{k+2}$ by $'X^{k+1}$: $('A^{k+2})^{\bullet} = {}'X^{k+1}$. It is easy to see that the cycles $'X^{k+1}$ thus defined form a system of base cycles of the $(k + 1)$-dimensional Betti-group of Q^{r+1}, since the looping coefficients (in R^{r+1}) of the cycles $'X^{k+1}$ with the cycles of K are the same as the looping coefficients of X^k with the cycles of K (in R^r). The

[7] The same result will be obtained by applying the construction, since the intersection in this case does not depend on the chain bounded by X^k.

[8] G. Nöbeling "Ein dimensionstheoretischer Isotopiesatz," *Monatshefte für Math. und Phys.* B. 41, Heft 1, 1934.

[9] Cf. Lefschetz, "Manifolds with a boundary and their transformations," *Trans. Am. Math. Soc.*, 29 (1927), 429–462.

heights h for different cycles may have arbitrary values. We choose them different for every two cycles corresponding to (1).

Now it is easy to show that the rings of Q^r and Q^{r+1} are isomorphic. If X^k and Y^l are two base cycles of Q^r, their product $[X^k Y^l] = (A^{k+1} B^{l+1})^{\bullet}$. The product of the corresponding cycles $['X^{k+1}\ 'Y^{l+1}] = ('A^{k+2}\ 'B^{l+2})^{\bullet}$. But $'A^{k+2}\ 'B^{l+2}$ is a cylinder constructed on $A^{k+1} B^{l+1}$ in the way described, the height being the lesser of the heights of the cylinders $'A^{k+2}$ and $'B^{l+2}$. Therefore the product $['X^{k+2}\ 'Y^{l+2}] = ('A^{k+2} 'B^{l+2})$ is homologous to the cycle corresponding in R^{r+1} to the product $[X^k Y^l]$. This completes the proof that the ring of K is independent of the dimension of R^r.

We have seen that the rings of K obtained by imbedding K in different manners in a Euclidean space of sufficiently high dimension ($r \geqq 2n + 2$) are isomorphic. But the same is true when we imbed K in a Euclidean space of arbitrary (even low) dimension. In fact let the ring of K^n be defined uniquely when K^n is imbedded in different manners in R^r. If we now imbed it in different ways in R^{r-1} all the rings obtained are isomorphic with the ring defined by means of R^r and therefore are isomorphic among themselves.

It might be remarked that we have not used the invariant definition of the ring of Q in order to obtain the ring of K. The defined ring of a complex is its topological invariant. It is isomorphic with the ring of a complementary space when that ring exists. (We are sure of the existence of the ring in the complementary space only when $r \geqq 2n + 2$.) We might note that the elements of the ring of a complex are precisely the characters of the Betti groups of the complex.

In conclusion I wish to express my sincere thanks to Professor Pontrjagin for his many valuable suggestions and for the help he has given me in my work.

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MANIFOLDS WITH ABELIAN FUNDAMENTAL GROUPS

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1. In a recent paper we showed[1] that the rank[2] of the (abelian) fundamental group of an n-dimensional group manifold can not exceed n. At the time of writing that paper it seemed to the author that this theorem might hold for any manifold with an abelian fundamental group. This is of course true—at least for compact manifolds—when $n = 2$. We shall show below that it is also true for $n = 3$ but is *not* true in general when $n > 3$. In order to construct counter examples for $n > 3$ we shall first determine the fundamental group of the symmetric product k_{2n} of a finite complex K_n by itself.

2. Theorem. *The fundamental group of k_{2n} is abelian.*

Proof. Let K' be a copy of K and let x and x' denote corresponding points in K, K'. Let τ be the self-homeomorphism (of period 2) of the direct product $K \times K'$ defined by

$$\tau(x \times y') = y \times x'.$$

Let x_0 be a definitely chosen point of K. There can be chosen a set of generators ξ_i, η_i for the group of $K \times K'$ such that (1) the ξ's and η's are paths beginning and ending at $x_0 \times x_0'$; (2) the ξ's are commutative with the η's; (3) for each i, ξ_i and η_i are images of each other under τ.[3]

3. Let t be the (2-1) mapping of $K \times K'$ onto k defined by

$$t(x \times y') = (x, y) = (y, x) = \text{a point of } k.$$

Two points of $K \times K'$ which are images of each other under τ correspond under t to one and the same point of k. Hence

$$t(\xi_i) = t(\eta_i) = (\text{say})\zeta_i.$$

Here $t(\xi_i)$ means the path described by $t(p)$ as p describes a circuit on ξ_i in the positive direction. The ζ_i are paths in k beginning and ending at (x_0, x_0).

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[1] *The fundamental group of a group manifold*, these Annals, vol. 36, No. 1 (1935), 210–229.

[2] By the *rank* of an abelian group we mean the maximum number of linearly independent elements which it contains.

[3] Let ω_i be paths in K beginning and ending at x_0 and let ω_i' be their duplicates in K'. If the ω_i are a set of generators for the group of K, we may take $\xi_i = x_0 \times \omega_i'$ and $\eta_i = \omega_i \times x_0'$. See H. Künneth, *Zur Bestimmung der Fundamentalgruppe einer Produktmannigfaltigkeit*, Sitzungsberichte Erlangen, Vol. 54–55 (1924), 190–196.

Since t is continuous, a relation of the form $\Pi(\xi, \eta) = 1$ corresponds to the relation $\Pi(t(\xi), t(\eta)) = \Pi'(\zeta) = 1$, that is, to a relation among the ζ_i obtained from the original relation by replacing each ξ_i and η_i by ζ_i. In particular, from the relations $\xi_i\eta_i\xi_i^{-1}\eta_j^{-1} = 1$ we obtain $\zeta_i\zeta_j\zeta_i^{-1}\zeta_j^{-1} = 1$, so that in order to prove that the group of k is abelian, we need merely prove that the ζ's constitute a set of generators for it.

4. Let k_0 be the totality of points (x, x), $x \subset K$. It will be seen from the definition of t that if Z is an arbitrary point of k, $t^{-1}(Z)$ is continuous and single-valued over k_0. If Z is not in k_0, $t^{-1}(Z)$ has two distinct determinations which vary continuously with Z. If Z approaches a point Z_0 of k_0, the two determinations of $t^{-1}(Z)$ converge to $t^{-1}(Z_0)$. It follows from these remarks that if Z is constrained to vary along a path ζ' which begins and ends on k_0 but otherwise fails to meet k_0, there can be determined by continuity two continuous single-valued branches $t_1^{-1}(Z)$ and $t_2^{-1}(Z)$ of $t^{-1}(Z)$, defined over ζ'.

Let ζ be an arbitrary path in k beginning and ending at (x_0, x_0). k is a complex, and in fact possesses a triangulation relative to which k is a subcomplex.[4] Hence on performing a preliminary deformation of ζ if necessary, we may take ζ as consisting of a finite number of arcs, each of which either begins and ends on k_0 and otherwise fails to meet k_0, or else lies entirely in k_0. Let ζ_j' be the arcs of the first type and let $s(Z)$ equal a definitely chosen one of the two branches of $t^{-1}(Z)$ corresponding to each ζ_j', and elsewhere on ζ let $s(Z) = t^{-1}(Z)$. Then $s(Z)$ is continuous and single-valued on ζ and as Z describes ζ in the positive direction, $s(Z)$ describes a path ζ^* in $K \times K'$, beginning and ending at $x_0 \times x_0'$. Moreover, for each Z in ζ we have $t(s(Z)) = Z$; hence $t(\zeta^*) = \zeta$. Therefore, since ζ^* equals a product of the ξ_i's and η_i's, ζ will equal (§3) a product of the ζ_i's, which completes the proof.

5. Let G be a group and A its commutator group, and let G/A be denoted by G^*. G^* is the group G "made abelian".

THEOREM. *Let $G(k_{2n})$ and $G(K_n)$ be the fundamental groups of k_{2n} and K_n. Then $G(k_{2n}) = G^*(K_n)$.*

PROOF. Denoting Betti numbers by R, we have

$$R_1(k_{2n}) = R_1(K_n)$$

$$R_1(k_{2n}; 2^u) = R_1(K_n; 2^u)^5 \qquad (u = 1, 2, \cdots).$$

[4] See M. Richardson, *On the homology characters of symmetric products*, Duke Math. Jour., Vol. 1, No. 1 (1935), 50–69.

[5] The first relation follows from M. Richardson, loc. cit., Theorem 2, p. 58. The second relation follows when $u > 1$ from Theorem 6 of the same paper and for $u = 1$, from the formulas on p. 618 of the author's paper, *The topology of involutions*, Proc. Nat. Acad., Vol. 19, No. 6 (1933), 612–618. In deriving the stated relations from the theorems just cited, the 1-dimensional Betti numbers of $K \times K'$ must be known. For their determination, see Lefschetz, *Topology*, p. 228.

Hence $B_1(k_{2n}) = B_1(K_n)$, where B stands for the complete 1-dimensional Betti group.[6] Now $B_1(K_n) = G^*(K_n)$ and $B_1(k_{2n}) = G^*(k_{2n}) = G(k_{2n})$.[7] Hence $G(k_{2n}) = G^*(K_n)$.

6. Let K_2^p be a closed orientable surface of genus p. The symmetric product k_4^p is a manifold,[8] and from the theorem just proved, $G(k_4)$ is a free abelian group of rank $2p$.[9] Let κ_0 be a point, κ_1 a circle, and $\kappa_n (n \geqq 2)$ an n-torus. Let h_3 be a 3-sphere and let

$$\mathfrak{k}_m^p = k_4^p \times \kappa_{m-4} \qquad (m \geqq 4, p \geqq 0)$$

$$\mathfrak{h}_m^p = k_4^p \times \kappa_{m-7} \times h_3 \qquad (m \geqq 7, p \geqq 0).$$

The $\mathfrak{k}_m$'s and $\mathfrak{h}_m$'s are compact m-dimensional manifolds and since the fundamental group of a direct topological product is the direct product of the factors,[10] it follows that $G(\mathfrak{k}_m^p)$ and $G(\mathfrak{h}_m^p)$ are free and abelian, and that

$$\text{(R)} \qquad \begin{aligned} \operatorname{rank} G(\mathfrak{k}_m^p) &= 2p + m - 4 \qquad (m \geqq 4) \\ \operatorname{rank} G(\mathfrak{h}_m^p) &= 2p + m - 7 \qquad (m \geqq 4). \end{aligned}$$

For a given $m \geqq 7$, the two sets of integers $2p + m - 4$ and $2p + m - 7$ $(p = 0, 1, \cdots)$ taken together, include all integers $\geqq m$. It follows from this and from the first relation (R) that *for every pair of integers m, $q (m \geqq 7, q \geqq m)$, there exists a compact m-dimensional manifold whose fundamental group is free, abelian, and of rank q; for each $m \geqq 4$ the same is true if the phrase "of rank q" is replaced by "of rank $> q$."* We leave open the question whether or not the first part of this theorem holds for $m = 4, 5, 6$.

7. Let M be a finite (i.e. compact) 3-dimensional topological manifold,[11] and assume that its fundamental group $G(M)$ is abelian.

THEOREM. *The rank of $G(M)$ is $\leqq 3$.*

PROOF. Let Γ be the universal covering space of M. $G(M)$ can be realized by a group of self-homeomorphisms $T_g(g \subset G(M))$ of Γ. The T_g's are the "deck-transformations" of Γ. Γ is a manifold and has a triangulation—namely that induced by the triangulation of M—which is preserved under the T_g's. If E is a point set in Γ, we shall denote $T_g E$ by E_g.

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[6] A. W. Tucker, *Modular homology characters*, Proc. Nat. Acad., Vol. 18, (1932), 467–471.

[7] Seifert and Threlfall, *Lehrbuch der Topologie*, p. 173.

[8] It is shown in Veblen and Young, *Projective Geometry*, Vol. II, pp. 270–271, that when $p = 0$, k_4^p is homeomorphic to the complex projective plane π. A simple modification of the argument shows that in any case, k_4^p has *locally* the same structure as π.

When $n > 2$ and K_n is a manifold, k_{2n} is a pseudomanifold, *but not a manifold.* This follows from certain results in the author's paper, cited in footnote 5.

[9] That is, generated by $2p$ generators which satisfy no relations other than those which assert that the generators are commutative.

[10] Seifert and Threlfall, loc. cit., p. 156.

[11] Lefschetz, *Topology*, p. 119.

8. Let X be a finite sum of 3-cells of Γ such that $\Gamma = \sum_g \bar{X}_g$ and $X_g \cap X_h = 0$ whenever $g \neq h$.[12] Now $G(M)$ is the direct sum of a finite group $\mathfrak{g}_0$ and a free group $\mathfrak{g}$ so that every g is of the form $\lambda_0 + \lambda$ where $\lambda_0 \subset \mathfrak{g}_0$, $\lambda \subset \mathfrak{g}$. Let $\Omega = \sum_{\lambda_0} X_{\lambda_0}$. Then $\sum_\lambda \bar{\Omega}_\lambda = \sum_{\lambda_0,\lambda} \bar{X}_{\lambda_0+\lambda} = \Gamma$ and moreover, $\Omega_\lambda \cap \Omega_\mu = 0$ whenever $\lambda \neq \mu$ $(\lambda, \mu \subset \mathfrak{g})$. Each Ω_λ is a finite sum of 3-cells and it is obvious that the closures of at most a finite number of distinct Ω_λ's can meet the closure of any given one of them.

9. We can think if $\mathfrak{g}$ as a group of vectors λ with integer components. Thus we shall write $\lambda = (\lambda_1, \cdots, \lambda_h)$ where the λ_i's are integers. To assume that the rank of $G(M)$ is > 3 is equivalent to assuming that $h > 3$. We shall make this assumption and obtain a contradiction.

Let x be a definitely chosen vertex in $\bar{\Omega}$. Vertices of the form x_λ[13] will be called *lattice points*. The *coördinates* of x_λ are $\lambda_1, \cdots, \lambda_h$ and the *distance* $d(x_\lambda, x_\mu)$ will be defined by the ordinary euclidean formula.

10. Let $\lambda^1, \lambda^2, \cdots$ be a definite ordering of the vectors λ and let $\Omega^n = \sum_{i=1}^n \bar{\Omega}_{\lambda^i}$. For every compact set E there is a k such that $E \subset \Omega^k$. Moreover, *there exists a number δ such that* $\bar{\Omega}_\mu \cap \bar{\Omega}_\nu = 0$ whenever $d(x_\mu, x_\nu) > \delta$. For if not, we could choose sequences $\Omega_{\mu^i}, \Omega_{\nu^i}$ $(i = 1, 2, \cdots)$ with $d(x_{\mu^i}, x_{\nu^i}) \to \infty$ and $\bar{\Omega}_{\mu^i} \cap \bar{\Omega}_{\nu^i} \neq 0$. We then have $\bar{\Omega} \cap \bar{\Omega}_{\mu^i - \nu^i} \neq 0$ and since $d(x, x_{\mu^i - \nu^i}) = d(x_{\mu^i}, x_{\nu^i}) \to \infty$, it follows that infinitely many of the vectors $\mu^i - \nu^i$ are distinct, hence infinitely many of the sets $\bar{\Omega}_{\mu^i - \nu^i}$ are mutually exclusive; but the closures of these sets can not all meet $\bar{\Omega}$ (§8), hence our assertion is established.

11. In the following, *all chains and bounding relations are modulo* 2. Let $\epsilon^1 = (1, 0, \cdots 0), \cdots, \epsilon^h = (0, \cdots, 0, 1)$. We shall show that finite chains u^i, v^{ij}, w^0 $(i, j, = 1, \cdots h; i < j)$, of dimensions 1, 2, 3 can be chosen in Γ such that

$$(1)\quad u^i \to x + x_{\epsilon^i}$$

$$(2)\quad v^{ij} \to u^i + u^j + u^i_{\epsilon^j} + u^j_{\epsilon^i}$$

$$(3)\quad w^0 \to v^{23} + v^{13} + v^{12} + v^{23}_{\epsilon^1} + v^{13}_{\epsilon^2} + v^{12}_{\epsilon^3}.$$

The u's exist because Γ is connected. By means of (1) we see that the right sides of (2) are cycles. For example, $u^1_{\epsilon^2} \to (x + x_{\epsilon^1})_{\epsilon^2} = x_{\epsilon^2} + x_{\epsilon^1+\epsilon^2}$ and similarly $u^2_{\epsilon^1} \to x_{\epsilon^1} + x_{\epsilon^1+\epsilon^2}$. Hence the sum of the boundaries of $u^1 + u^2 + u^1_{\epsilon^2} + u^2_{\epsilon^1}$ is 0. Since Γ is simply connected, the v's exist, and by means of (2) we see that the right sides of (3) are cycles. To show the w^0 exists we first prove that *every infinite 1-cycle of Γ is ~ 0.*

Let c be such a cycle. We may assume that no proper subset of the 1-cells

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[12] See pp. 212–213 of the paper cited in footnote 1 for a construction for X (X is the same as the set Ω_0 defined on p. 213).

[13] From now on, the greek letters λ, μ, ϵ, etc. will refer to elements of $\mathfrak{g}$.

which occur in c forms a cycle, since in general c would be a sum of such irreducible cycles. Thus $\bar{c}$ is an infinite non-singular polygonal line, say

$$c = \cdots | p^{-1} p^0 | + | p^0 p^1 | + \cdots$$

where $| p^i p^{i+1} |$ is the 1-cell whose ends are p^i, p^{i+1}.

Let (p, q) denote an arbitrary 1-chain with boundary $p + q$. Let x_λ and $x_{\lambda'}$ be *adjacent* lattice points, that is, such that $\lambda' - \lambda = \pm \epsilon^i$. Let $\{x_\lambda x_{\lambda'}\}$ be u_λ^i or $u_{\lambda'}^i$, according as $\lambda' - \lambda = \epsilon^i$ or $-\epsilon^i$. In either case $\{x_\lambda x_{\lambda'}\} \rightarrow x_\lambda + x_{\lambda'}$. Let $[x_\lambda x_\mu]$ be an arbitrary chain of the form $\{x_\lambda x_{\lambda'}\} + \{x_{\lambda'} x_{\lambda''}\} + \cdots + \{x_\tau x_\mu\}$; its boundary is $x_\lambda + x_\mu$.

Let $\bar{\Omega}_\alpha, \cdots, \bar{\Omega}_\gamma$ be the $\bar{\Omega}_\lambda$'s which meet $\bar{\Omega}$. Let $|pq|$ be a 1-cell and x_λ a lattice point, both contained in $\bar{\Omega} + \bar{\Omega}_\alpha + \cdots + \bar{\Omega}_\gamma$. For each of the finite number of ways of choosing p, q, λ, choose *definite* finite chains (xp), $(x_\lambda q)$, $[xx_\lambda]$ and for each cycle $|pq| + (xp) + (x_\lambda q) + [xx_\lambda]$ formed from these chains, choose a definite finite 2-chain with that cycle for its boundary. Choose N so large that the closures of all the cycles and chains just described lie in Ω^N.

Returning to c, choose for each i, a definite $\bar{\Omega}_{\mu^i}$ containing p^i. Clearly $d(x, x_{\mu^i}) \rightarrow \infty$ as $|i| \rightarrow \infty$. Hence at most a finite number of vectors of the set $\lambda^n + \mu^i$ $(n = 1, \cdots h; i = 1, 2, \cdots)$ can be equal. Consequently at most a finite number of the corresponding sets $\Omega_{\lambda^n + \mu^i}$ can be identical, and hence (§8) at most a finite number of the closures of these sets can have a non-zero intersection and hence finally, *at most a finite number of the sets* $\Omega_{\mu^i}^N = \sum_{n=1}^h \bar{\Omega}_{\lambda^n + \mu^i}$ $(i = 1, 2, \cdots)$ *can have a non-zero intersection.*

Since $\bar{\Omega}_{\mu^{i+1}}$ meets $\bar{\Omega}_{\mu^i}$, it follows from the definition of N in the next to last paragraph that for each i, there can be chosen definite chains $[x_{\mu^i} x_{\mu^{i+1}}]$, $(x_{\mu^i} p^i)$, $(x_{\mu^{i+1}}, p^{i+1})$, f^i, which are images under T_{μ^i} of chains in Ω^N and which satisfy the relation

$$f^i \rightarrow | p^i p^{i+1} | + (x_\mu^i p^i) + (x_{\mu^{i+1}} p^{i+1}) + [x_{\mu^i} x_{\mu^{i+1}}] = b^i \text{ (say)}.$$

Since $b^i \subset \Omega_{\mu^i}^N$, $f^i \subset \Omega_{\mu^i}^N$ and since at most a finite number of the $\Omega_{\mu^i}^N$'s have a non-zero intersection, it follows that the sums

$$\cdots + f^{-1} + f^0 + f^1 + \cdots, \qquad \cdots + b^{-1} + b^0 + b^1 + \cdots,$$

(as well as the sum written below) are well defined chains, the second being the boundary of the first. The second sum equals $c + d$, where

$$d = \cdots + [x_{\mu^{-1}} x_{\mu^0}] + [x_{\mu^0} x_{\mu^1}] + \cdots.$$

Since $c \sim d$, we need only prove that $d \sim 0$.

The transformations T_λ correspond in an obvious manner to a group of translations in a cartesian S_h. Moreover if we think of each u^i as corresponding to the unit segment at the beginning of the positive x_i-axis in S_h and v^{ij} as corresponding to the unit square subtended by the i^{th} and j^{th} segments, then each chain sum of images of x, or of the u's or of the v's corresponds in S_h to a definite chain sum of the corresponding images of the origin or of the corresponding unit

segments or unit squares. This correspondence between chains is (1, 1) and, as will be seen from the nature of the boundary relations (1) and (2), preserves boundary relations. Now from the definition of the chains [ ], d is of the form $\cdots + \{x_{\nu-1}x_{\nu_0}\} + \{x_{\nu_0}x_{\nu_1}\} + \cdots$ in which each component chain is an image of one of the u's. The corresponding chain in S_h is an infinite cycle and bounds a 2-chain sum of images of the unit squares. The sum of the corresponding transforms of the corresponding v's is a 2-chain with d for its boundary. Hence $d \sim 0$.

It follows that if Λ is the total ideal set of Γ, then $R_1(\Gamma; \Lambda) = 0$. Hence $R_2(\Gamma - \Lambda) = 0$,[14] and hence the existence of w^0 is established.

12. Let

$$(4) \qquad u = u^1, \qquad v = \sum_{-\infty}^{\infty} v_{(m, \dots)}, \qquad w = \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} w^0_{(m, n, \dots)}$$

$$(5) \qquad U = \sum_{-\infty}^{\infty} u_{(n, \dots)}, \qquad V = \sum_{-\infty}^{\infty} v_{(0, n, \dots)}, \qquad W = \sum_{-\infty}^{\infty} w_{(0, 0, n, \dots)},$$

where the dots represent zeros. It is clear that all the sums are well defined chains. Moreover, by (1), (2), (3), (4) we see that U, V, W are cycles. We shall show successively that $U \neq 0$, $V \neq 0$, $W \neq 0$.

Let $U^+ = \sum_0^{\infty} u_{(n, \dots)}$, $U^- = \sum_{-1}^{-\infty} u_{(n, \dots)}$ so that $U = U^+ + U^-$. We can choose a k so large that $x_{(k, \dots)}$ is not contained in $\overline{U}^-$. For let q be so large that $\bar{u} \subset \Omega^q$. Then $\overline{U}^- \subset \sum_{n=-1}^{-\infty} \Omega^q_{(n, \dots)}$. There is an upper bound for the first coördinates of the lattice points contained in the set represented by the last sum; hence their distances from $x_{(k, \dots)}$ will all be $> \delta$ (see §10) if k is sufficiently large. Such a k has the desired property.

Let $U^* = \sum_0^k u_{(n, \dots)}$. Then $U^* \to x + x_{(k, \dots)}$, whereas $U^+ \to x$. Hence an odd number of cells of U^* and an equal number of cells of $U^* + U$ abut at $x_{(k, \dots)}$. Since these two sets of cells are mutually exclusive and since no cell of either set occurs in U^- (by choice of k), it follows that each of these cells occurs in the symbol of $U^- + U^+ = U$ with the coefficient $+1 \cdots$. Hence $U \neq 0$.

By similar reasoning, $V \neq 0$. First, k can be chosen so that $U_{(0, k, \dots)}$ does not meet $\bar{V}^-$ where $V^- = \sum_{-1}^{-\infty} v_{(0, n, \dots)}$. To see that such a k exists, choose q, q' such that $\bar{u} \subset \Omega^q$, $\bar{v}^{12} \subset \Omega^{q'}$. Then

$$\overline{U}_{(0, k, \dots)} \subset \sum_{n=-\infty}^{\infty} \Omega^q_{(n, k, \dots)}, \qquad \bar{V}^- \subset \sum_{n=-1}^{-\infty}\sum_{m=-\infty}^{\infty} \Omega^{q'}_{(m, n, \dots)}.$$

There is an upper bound for the second coördinates of the lattice points which are contained in the second sum. Hence by choosing k sufficiently large, all the lattice points which are contained in the first sum will be at distances $> \delta$ from each of those in the second, and such a k has the desired property. Now let $V^* = \sum_1^k v_{(0, n, \dots)}$. Then since

[14] Lefschetz, *Topology*, p. 314.

$$(6)\quad v \to \sum_{m=-\infty}^{\infty} (u^1_{(m,\cdots)} + u^2_{(m,\cdots)} + u^1_{(m,1\cdots)} + u^2_{(m+1,\cdots)}) = U + U_{(0,1,\cdots)})$$

it follows that $V^* \to U + U_{(0,k,\cdots)}$. Since $U_{(0,k,\cdots)}$ is an image of U, it is $\neq 0$ and hence contains at least one 1-cell σ with coefficient $+1$. As above, an odd number of 2-cells from each of the chains V^* and $V^* + V$ abut at σ and hence, as above $V \neq 0$. Using this result, it is proved in an exactly similar manner that $W \neq 0$.

Now from (3), (4), (5) we find that $w \to V + V_{(0,0,1,\cdots)}$ (cf. (6)). Hence by (5) $W \to \sum_{-\infty}^{\infty} V_{(0,0,n,\cdots)} + V_{(0,0,n+1,\cdots)} = 0$ and hence W is a 3-cycle. Since $W \neq 0$ and $R^3(\Gamma; \Lambda) = 1$, it follows that $\overline{W} = \Gamma$. On the other hand, $W = \sum_q\sum_m\sum_n w^0_{(q,m,n,\cdots)}$. Hence, as above, for k sufficiently large, $\Omega_{(0,0,0,k,\cdots)}$ fails to meet $\overline{W}$, which is a contradiction.

REMARKS. 1. The papers referred to in §§2–5 for the proof of the theorem of §5, deal only with finite complexes. It is not difficult, however, to show that the theorem holds also for infinite complexes.

2. For every $m > 1$ and $q \geqq m$ there exists a complex K with a free abelian fundamental group of rank q; in fact we may take for K a q-torus, triangulated, from which all the cells of dimension $> m$ have been removed.[15] Whether or not the theorem of §7 holds for complexes which are not manifolds but, say, pseudo-manifolds, we do not know. The manifold requirement was used in order to apply the Lefschetz duality formulas (§11).

3. The theorem of §7 also holds when M is an infinite manifold, the proof being essentially the same as for the finite case. The bounding relation (3) is to be taken modulo L_1, where L_1 is that ideal set of Γ which "covers" the total ideal set of M. If L_2 is the complementary ideal set,[16] the existence of w^0 (no longer finite) follows from the fact that $R_2(\Gamma - L_2; L_1) = 0$, proved by first showing that $R_1(\Gamma - L_1; L_2) = 0$ and then applying the Lefschetz duality formulas (loc. cit.).

4. A short and elementary proof of our theorem for group manifolds (§1) can be made, briefly, as follows: let M be a continuous group of say 3-dimensions. Assume that M contains independent paths $\omega_1, \cdots, \omega_h (h > 3)$ beginning and ending at (say) θ, each of infinite order. Let $\sum_0, \cdots, \sum_3$ be the total subcomplexes of dimensions $0, \cdots, 3$ contained in a subdivision of S_h into unit "cubes". Let θ_i be an arbitrary point of ω_i and let

$$\omega_{ij}\ (i < j) \text{ be the totality of points } \theta_i\theta_j,$$

$$\omega_{ijk}\ (i < j < k) \text{ be the totality of points } \theta_i\theta_j\theta_k$$

where the products are group products defined for M. Each ω_{ijk} is a singular image of a 3-torus, and it is easy to see that the totality of these sets is covered in Γ by a set $\sum_3^*$ which is a singular image of $\sum_3$, say $\sum_3^* = \tau \sum_3$. τ simul-

[15] Cf. Seifert and Threlfall, loc. cit., p. 162.
[16] Lefschetz, loc. cit., p. 297.

taneously maps $\sum_2$, $\sum_1$, $\sum_0$ onto $\sum_2^*$, $\sum_1^*$, $\sum_0^*$, the sets which cover the ω_{ij}'s, the ω_i's and the point θ. τ can be chosen in such a way that if x is the point which corresponds to the origin in S_h, x_λ will correspond to the point in S_h with coördinates $\lambda_1, \cdots, \lambda_h$. It follows that for u^i in §11, we can take the image under τ of the unit segment at the beginning of the positive x_i-axis and for v^{ij} the image of the unit square subtended by the x_i and x_j unit segments and for w^0 the image of the unit cube subtended by the x_1, x_2, and x_3 unit segments. The resulting 1-, 2-, 3-chains may not be simplicial but can be deformed into simplicial ones without affecting the character of the bounding relations. We then proceed as in §12 to a contradiction. This proof is, in principle, perfectly general as regards the number of dimensions.[17]

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[17] Hurewicz has recently obtained the theorem in question as a consequence of his very general and far reaching researches on the topology of deformations. See the Proceedings of the Amsterdam Academy, Vol. XXXIX, No. 2 (1936), p. 223.

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TWO-DIMENSIONAL MANIFOLDS AND TRANSITIVITY

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In a recent paper, Morse (2) has proved the existence of transitive geodesics on any closed orientable surface, S, of genus greater than one, provided the geodesics on S are uniformly unstable. The geodesics are studied conveniently by mapping the universal covering surface of S into the interior of the unit circle. It is not difficult to show the transitivity of a non-denumerable set of geodesics provided the geodesics behave in the large like hyperbolic lines and are in one-to-one correspondence with the hyperbolic lines. The hypothesis of unicity is then said to be satisfied. The hypothesis of uniform instability is sufficient to imply this behavior of the geodesics.

But the hypothesis of unicity is not necessary for transitivity. To prove the existence of transitive geodesics it is sufficient to assume in place of the one-to-one correspondence of the geodesics and hyperbolic lines, a one-to-one correspondence of the geodesic rays and hyperbolic rays with the same initial point interior to the unit circle. The hypothesis of uniform instability can be weakened and still imply this. It is one of the objects of this paper to develop this result.

Fuchsian groups again play an important rôle. Considerable use is made of known results concerning the transitivity of hyperbolic lines. There is no object in restricting the Fuchsian groups under consideration to groups with a finite set of generators or to groups which have a fundamental region lying entirely interior to the unit circle. The proof of the stated result holds for all Fuchsian groups for which there exist transitive hyperbolic lines. But the existence of transitive hyperbolic lines is known in the case of any Fuchsian group of the first kind. These are the groups which cease to be properly discontinuous at all points of the unit circle. Thus there is proved the existence of transitive geodesics on certain surfaces of infinite genus and certain surfaces with singularities.

But the result does more than merely extend the kinds of surfaces for which transitivity can be proved. It means replacing topological transformations by conformal transformations. This seems desirable if the methods used in proving metric transitivity in the case of constant negative curvature are to carry over to the non-constant case.

Finally, if the geodesics do have the property of unicity, known results, notably those of Myrberg, concerning the hyperbolic lines carry over immediately to the geodesics. It is found that the intransitive geodesic rays through a point interior to the unit circle form a non-denumerable set and that the points of the unit circle which determine transitive geodesic rays form a set of linear measure 2π.

1. **Uniform instability.** Let U be the circle $u^2 + v^2 = 1$, and let Ψ be its interior. The metric in Ψ will be defined by

$$ds^2 = \frac{4f^2(u, v)(du^2 + dv^2)}{(1 - u^2 - v^2)^2} \tag{1.1}$$

where $f(u, v)$ is of class C^5 and $0 < a \leqq f(u, v) \leqq b$ in Ψ. Given a curve segment, γ, of Class C' in Ψ, its *length*, $L(\gamma)$, is defined as $\int ds$ evaluated over the given curve with ds defined by (1.1). The term *geodesic* will refer to the geodesics defined by (1.1).

If, in particular, $f(u, v) \equiv 1$, in Ψ, the geodesics are arcs of circles orthogonal to U and will be referred to as *hyperbolic lines* or *H-lines*. Given two points, P_1 and P_2, in Ψ, there is a unique segment of an H-line joining these points and $\int ds$, evaluated over this segment, where ds is given by (1.1) with $f(u, v) \equiv 1$, will define the *hyperbolic distance*, $H(P_1, P_2)$, between P_1 and P_2. The *hyperbolic length*, $H(\gamma)$, of a curve segment, γ, of Class C' will be defined as was length but with the restriction $f(u, v) \equiv 1$.

Assuming no longer that $f(u, v) \equiv 1$ in Ψ, the question arises as to whether there is a minimizing geodesic in Ψ joining two points of Ψ. This is answered by the following theorem, which follows readily from well known results on absolute minima.

THEOREM 1.1. *If P_1 and P_2 are points in Ψ, there exists a geodesic segment joining P_1 and P_2 such that the length of this segment is not greater than that of any other segment of Class C' in Ψ joining P_1 and P_2. Defining the distance, $D(P_1, P_2)$, between P_1 and P_2 as the length of such a geodesic segment, the following inequalities hold*:

$$a \cdot H(P_1, P_2) \leqq D(P_1, P_2) \leqq b \cdot H(P_1, P_2).$$

For let U_r be a circle with center at the origin and of radius r, $0 < r < 1$, so chosen that both P_1 and P_2 are interior to U_r and such that the hyperbolic distance between either P_1 or P_2 and an arbitrary point of U_r is greater than $a^{-1} \cdot b \cdot H(P_1, P_2)$. If we let Ψ be $\mathbf{R}$ and U_r and its interior be $\mathbf{R}_0$, the conditions of Hilbert's theorem (Bolza, pp. 421–422) concerning absolute minima are satisfied and there is a minimizing rectifiable curve, γ_1, in $\mathbf{R}_0$ joining P_1 and P_2. This curve can have no point on U_r. For if this were the case, let Q be the first point of γ_1 on U, tracing γ_1 in the sense from P_1 to P_2. The segment P_1Q of γ_1 would be a geodesic segment γ. The following inequalities hold:

$$L(\gamma_1) > L(\gamma) \geqq a \cdot H(\gamma) \geqq a \cdot H(P_1, Q) > a \cdot a^{-1} \cdot b \cdot H(P_1, P_2) = bH(P_1, P_2).$$

But if γ_2 denotes the H-line segment joining P_1 and P_2, we have $L(\gamma_2) \leqq b \cdot H(P_1, P_2)$, and the H-line segment joining P_1 and P_2 has length less than that of the minimizing curve γ_1. This contradiction implies that γ_1 must lie entirely interior to U_r and hence is entirely a geodesic. It is evidently a minimizing curve with respect to all rectifiable curves in Ψ joining P_1 and P_2.

The inequalities of the theorem are again obvious consequences of the inequalities imposed on $f(u, v)$.

The terms *Class A* and *type* will be used here as defined by Morse (2, pp. 53–54), with R-distance replaced by distance as defined above. Certain results of Morse hold here with slight modification of their proofs.

I. (Morse (1), p. 41). *There exists a positive constant, D, determined by $f(u, v)$ of (1.1), such that in Ψ no segment of a geodesic of Class A can recede an H-distance greater than D from the H-segment joining its end points.*

II. (Morse (1), p. 44). *Corresponding to any H-line, there exists at least one unending Class A geodesic of the same type. Conversely, every geodesic of Class A is of the type of some H-line.*

III. (Morse (2), p. 54). *Corresponding to each H-ray issuing from a point P in Ψ there exists a geodesic ray of Class A of the same type. Conversely, every Class A geodesic ray issuing from P is of the type of some H-ray. There exists a constant K, determined by $f(u, v)$ of (1.1), such that the type distance between two geodesic rays of Class A of the same type or a geodesic ray of Class A and an H-ray of the same type never exceeds K.*

Let g be an oriented geodesic in Ψ and let s be the arc-length on g measured from some point P_0 of g. In the Jacobi differential equation

$$\frac{dw^2}{ds^2} + K(s)\cdot w = 0, \tag{1.2}$$

of g, $K(s)$ is the Gaussian curvature determined by (1.1) at the point s of g.

Definition 1.1. *The geodesic g is unstable at P_0 with instability function $I(s)$, if there exists a continuous functin $I(s)$ with the properties:*

1. $I(s)\cdot s > 0,\ s \neq 0$;
2. $\lim_{s\to+\infty} I(s) = +\infty,\ \lim_{s\to-\infty} I(s) = -\infty$;
3. *If $w(s)$ is the solution of (1.2) for which $w(0) = 0$, $w'(0) = 1$, then $w(s) \geqq I(s)$, $s > 0$, and $w(s) \leqq I(s)$, $s < 0$.*

Definition 1.2. *The geodesics in Ψ are uniformly unstable with instability function $I(s)$ if each geodesic is unstable at each point with the same instability function, $I(s)$, for all points of all geodesics.*

If the geodesics in Ψ satisfy the condition of uniform instability there are no mutually conjugate points on any geodesic, the geodesics through a point of Ψ form a field in Ψ except at P and all geodesics are of Class A. Each geodesic is then of the type of an H-line and a geodesic ray with initial point in Ψ can have just one limit point (in the euclidean sense) on U. This point will be called the *point at infinity* of the given ray.

Theorem 1.2. *If the geodesics in Ψ are uniformly unstable, P is an arbitrary point of Ψ and Q is an arbitrary point of U, there is just one geodesic ray with initial point P and with Q as point at infinity.*

Suppose that there are two distinct geodesic rays, r and r', with P as initial point and Q as point at infinity. These rays, together with the point Q divide Ψ into two parts and the part interior to these rays together with these rays

will be denoted by Ψ_R. The geodesic rays, R, with initial point P and entering Ψ_R at P lie entirely in Ψ_R and form a field in Ψ_R.

Since r and r' are assumed to have the same point at infinity, Q, the distance from any point of one to some point of the other never exceeds K. The same is true of all pairs of geodesic rays in the set R. Corresponding to any point A of r, there exists a point A' of r' such that the length of the geodesic segment AA' is not greater than K. It is easily shown that this segment AA' must lie entirely in Ψ_R and hence must intersect all the geodesic rays in the set R, each in just one point.

Let β, $0 < \beta < 2\pi$, be the angle between r and r' at P, β measured in Ψ_R. From property 2 of $I(s)$, there exists an X such that $I(x) > K/\beta$, $x \geqq X$. Now let A be so chosen on r that the length of the segment PA of r is greater than $X + K$. Then any geodesic ray, $\bar{r}$, of the set R must intersect the geodesic segment AA' in a point $\bar{A}$ such that the distance $P\bar{A}$ is greater than X. For if this were not the case it would be possible to join A to P by a broken geodesic the total length of this being not greater than $X + K$, contrary to the assumption about the geodesic distance from P to A. Thus all the points of the segment AA' are at geodesic distance greater than X from P.

Let the angle β be divided into two equal angles by the ray r_1. This ray intersects the geodesic segment AA' in a point A_1. One of the segments into which A_1 divides AA' must be of length less than or equal to $K/2$. If this is AA_1, let the same process of division by a ray r_2 into two segments be applied to AA_1 and one of the segments of AA_1 thus obtained will be of length less than or equal to $K/2^2$. Continuing this process we obtain a sequence of pairs of geodesic rays with initial points at P, each obtained from the preceding by bisecting the angle at P, and such that if the angle at P formed by a pair is $\beta/2^n$, the segment cut off on AA' is not greater than $K/2^n$. These geodesic rays converge to a ray $\bar{r}$. Let $\bar{A}$ be the point in which $\bar{r}$ meets AA'. The geodesic distance $P\bar{A}$ is greater than X.

Let (u, φ) be geodesic polar coördinates with P as center. Since the geodesics through P form a field in Ψ, these geodesic polar coördinates give a one-to-one representation of Ψ with the exception of P itself. In these coördinates the element of arc-length (1.1) is given by

$$ds^2 = du^2 + G^2(u, \varphi)d\varphi^2, \tag{1.3}$$

where $G(u, \varphi)$ is non-negative and of at least Class C' for $u \geqq 0$, φ arbitrary, and the following conditions are satisfied:

$$G(u, \varphi) > 0,\ u > 0; \qquad G(0, \varphi) = 0; \qquad G_u(0, \varphi) = 1; \tag{1.4}$$

and

$$\frac{\partial^2 G}{\partial u^2} + K(u, \varphi)\cdot G = 0. \tag{1.5}$$

From (1.5), the function of u obtained from $G(u, \varphi)$ by holding φ constant is a solution of (1.2) which satisfies the boundary conditions $G(0, \varphi) = 0$, $G_u(0, \varphi) = 1$. From the hypothesis of uniform instability $G(u, \varphi) \geqq I(u)$, $u > 0$.

In these coördinates the geodesic segment AA' is given by $u = u(\varphi)$, of Class C' in $\varphi_1 \leqq \varphi \leqq \varphi_2$. If s denotes the length on AA' measured from some fixed point such that it increases with φ, we have, along AA',

$$\frac{ds}{d\varphi} = \sqrt{\left(\frac{du}{d\varphi}\right)^2 + G^2(u, \varphi)} \geqq G(u, \varphi) > \frac{K}{\beta}.$$

This holds in particular at $\bar{A}$. Denoting by $\bar{\varphi}$ the angle at P determining $\bar{r}$, hence $\bar{A}$, since $(ds/d\varphi)_{\varphi=\bar{\varphi}} > K/\beta$ and $ds/d\varphi$ is continuous, there exists a $\delta > 0$ such that $ds/d\varphi > K/\beta$ in the interval $\bar{\varphi} - \delta \leqq \varphi \leqq \bar{\varphi} + \delta$. But it has already been seen that there exists an infinite sequence of pairs of geodesic rays converging to $\bar{r}$ with the ratio of the length of the arc cut off on AA' to the angle between the pairs at P not greater than K/β. Eventually these pairs satisfy the condition $\bar{\varphi} - \delta \leqq \varphi \leqq \bar{\varphi} + \delta$ and we have a contradiction.

2. **Fuchsian groups and transitivity of the hyperbolic lines.** It will be assumed in this paragraph that $f(u, v) \equiv 1$ in Ψ. In this case the curvature of the differential form (1.1) is -1 and a well-known hyperbolic geometry is defined. The geodesics are now hyperbolic lines.

Let F be a Fuchsian group of the first kind with principal circle U (Ford, p. 68). The transformations of this group are linear fractional transformations taking Ψ into Ψ, U into U and (1.1) (with $f(u, v) \equiv 1$) is invariant under all such transformations.

Two point sets, within or on U, such that there is a transformation of F taking one into the other will be called *congruent*. Either will be said to be a *copy* of the other.

A directed H-line, AB, is *transitive* if, given an arbitrary ordered pair of intervals, I_1 and I_2 of U, there is a copy, $A'B'$, of AB, with A' congruent to A and in I_1, B' congruent to B and in I_2. If an oriented H-line is transitive, the same H-line with the opposite orientation is transitive. A directed hyperbolic ray, PA, P in and A on U, is *transitive* if, given any $\delta > 0$ and an arbitrary ordered pair of points, C and D, of U, there is a copy, $P'A'$ of PA with P' within euclidean distance δ of C and A' within euclidean distance δ of D. If a directed H-ray is transitive, the directed H-line of which it is a part is evidently transitive.

The question of the existence of transitive geodesics has been completely answered in the case of constant negative curvature. The following theorem is due to Koebe.

Theorem 2.1. (Koebe, p. 349). *If F is a Fuchsian group of the first kind with principal circle U, P is an arbitrary point interior to U and Q_1Q_2 is an arbitrary interval of U, there exists a transitive directed hyperbolic ray with initial point P and point at infinity in Q_1Q_2.*

3. **Instability and transitivity.** Again let F be a Fuchsian group of the first kind with principal circle U. A function defined in Ψ is *automorphic with respect to* F if it is invariant under all transformations of F. The restriction will no longer be imposed that $f(u, v) \equiv 1$, but that, in addition to the conditions of §1, *$f(u, v)$ must be automorphic with respect to F.* The quadratic form (1.1) is then *invariant under the transformations of F.*

It is convenient to introduce the space, E of elements. Let $\bar{E}$ denote the set of real triples (x, y, φ) where $x^2 + y^2 < 1$ and $0 \leqq \varphi < 2\pi$. Each such (x, y, φ) defines a point $P(x, y)$ of Ψ and a direction φ at that point, φ to be measured from the direction at P parallel to the positive real axis. Conversely, a point of Ψ and a direction at this point determines a point of $\bar{E}$. A transformation of F transforms $P(x, y)$ of Ψ into a point $P'(x', y')$ of Ψ and the direction φ at P into a direction φ' at P'. The points $p(x, y, \varphi)$ and $p'(x', y', \varphi')$ of $\bar{E}$ will then be said to be *congruent. The space E, of elements on the manifold under consideration, is obtained from the set $\bar{E}$ by identifying congruent points.*

To define neighborhoods in E, let e be a "point" of E. This point is a set (x_i, y_i, φ_i), $i = 1, 2, \cdots$, of congruent points of $\bar{E}$. Let (x_k, y_k, φ_k) be an arbitrary one of this set and δ an arbitrary positive number. Let S be the set of points (x, y, φ) of $\bar{E}$ which satisfy the inequalities

$$D(P, P_k) < \delta, \qquad |\varphi - \varphi_k + 2n\pi| < \delta, \text{ for some } n,$$

where P is the point (x, y) and P_k is the point (x_k, y_k), both in Ψ. Each such (x, y, φ) together with the points of $\bar{E}$ congruent to it defines a point of E. The totality of such points will define a *neighborhood, N_e, of e.*

It is easily seen that E, with neighborhoods thus defined, is a Hausdorff space. The terms open, closed, closure and limit point are defined in the usual way.

A directed curve, γ, of Class C' in Ψ defines a set of points, E_γ, of E, namely, the points of E defined by the points of the curve and the directions of the curve at these points. If the closure of E_γ is E, γ is *transitive.* It is easily seen that if $f(u, v) \equiv 1$, this definition coincides with that of transitivity given for hyperbolic lines.

Choosing an arbitrary point, e, of E, let (x, y, φ) be one of the set of congruent points of $\bar{E}$ defining e. This point (x, y, φ) determines a point $P(x, y)$ of Ψ and a direction φ at P, hence a directed geodesic g passing through P with direction φ there. Let s be the arc-length on g measured from P and taken in the positive direction on g. The arc s then determines a point P_s on g and the element of g at P_s determines a point e_s of E. Since e was arbitrary, a continuous transformation, $e \to e_s$, of E into itself is defined which is one-to-one. The totality of points e_s, s arbitrary, constitute the *stream-line determined by e.* Restricting s to non-negative values, the set of points is the *positive stream-line determined by e.*

LEMMA 3.1. *If λ is a non-singular continuous curve segment in E with the property that every segment of λ consisting of more than one point determines a set of positive stream-lines whose closure is E, every segment of λ has on it a point determining a transitive positive stream-line.*

The curve λ is the non-singular image of a line segment $0 \leqq t \leqq 1$, and its points will be denoted by the corresponding values of t.

It will be convenient to denote the "set of positive stream-lines determined by a segment of λ" *simply by the set of E determined by the segment.*

The neighborhoods in E which are defined by using only sets (x, y, φ) with rational arguments and only rational values of δ form a denumerable set N_1, $N_2, \cdots$. Since λ determines a set whose closure is E, there must be some point, t_1, of λ, which determines a set of E with a point in N_1. Then there exists a closed segment, λ_1, of λ, containing t_1 and such that each point of λ_1 determines a set with a point in N_1. From the hypothesis of the lemma the same argument can be applied to the interval λ_1 and N_2. Proceeding inductively, we obtain a sequence, $\lambda_1, \lambda_2, \cdots$, of closed intervals of λ, each contained in the preceding and such that λ_n, n any positive integer, has the property that each of its points determines a set of points of E which contains points in $N_1, N_2, \cdots, N_n$, inclusive. The intervals λ_i, $i = 1, 2, \cdots$, have at least one point, e_t, in common. But the set of E determined by e_t has points in every one of the neighborhoods N_i, $i = 1, 2, \cdots$, and e_t determines a transitive positive streamline.

In the preceding argument, instead of using λ initially, any subinterval could have been used, so that the point e_t could have been determined in any preassigned subinterval. This is the statement of the lemma.

THEOREM 3.1. *Let P be an arbitrary point of Ψ and Q_1Q_2 an arbitrary interval of U. If the condition of uniform instability is satisfied, there exists a transitive directed geodesic ray with initial point P and with point at infinity in Q_1Q_2.*

It can be assumed that the interval Q_1Q_2 is closed. For if this is not the case, by proving the theorem with Q_1Q_2 replaced by a closed subinterval the theorem will be proved.

If it is shown that, $Q_1'Q_2'$ being an arbitrary subinterval of Q_1Q_2, the set of directed geodesic rays with initial point P and point at infinity in $Q_1'Q_2'$ determine a set in E whose closure is E, the theorem will follow. For then the point P and the directions of these rays at P determine a curve λ in E and a properly chosen subsegment of λ will be non-singular and will satisfy the conditions of Lemma 3.1.

Consider the totality of geodesic rays with initial point P and points at infinity in $Q_1'Q_2'$ and let $E_{Q_1'Q_2'}$ be the set of points of E determined by all these rays. The closure of $E_{Q_1'Q_2'}$ is E if, e being an arbitrary point of E, there exists a sequence of points $e_1, e_2, \cdots$, of E with $\lim_{n\to\infty} e_n = e$. The point e is a set of points of $\bar{E}$ and let (x, y, φ) be one of these points. The point (x, y, φ) determines a directed geodesic, g, namely, that directed geodesic which passes through $R(x, y)$ and has direction φ there. Let A be the initial point at infinity of g, B the terminal point at infinity of g, and h the directed H-line AB. On AB there is point R' such that the hyperbolic distance $H(R, R')$ is less than D. Let $e_{p'}$ be the point of E determined by h at R'.

From Theorem 2.1 there exists an interior point Q of $Q_1'Q_2'$ such that the directed hyperbolic ray PQ is transitive. There exists a sequence, T_i,

$i = 1, 2, \cdots$, of transformations of F such that, in the euclidean sense, $\lim_{n\to\infty} T_n(P) = A$ and $\lim_{n\to\infty} T_n(Q) = B$. Let $P_n = T_n(P)$ and consider the sequence of directed geodesic segments P_nR. With increasing n the direction of P_nR at R must approach that of g at the same point. For otherwise a subsequence of the set P_nR, $n = 1, 2, \cdots$, could be chosen such that this sequence has as limiting geodesic ray a geodesic ray with end point R, point at infinity A and is not identical with the part AR of g. This contradicts Theorem 1.2. Let e_n be the point of E determined by P_nR at R. It has been shown that $\lim_{n\to\infty} e_n = e$.

It remains to show that there exists an N such that all e_n, $n > N$, belong to the set $E_{Q_1'Q_2'}$. The points $T_n^{-1}(R')$ have the property that, in the euclidean sense, $\lim_{n\to\infty} T_n^{-1}(R') = Q$. The point $T_n^{-1}(R)$ is within hyperbolic distance D of $T_n^{-1}(R')$, hence, again in the euclidean sense, $\lim_{n\to\infty} T_n^{-1}(R) = Q$. There exists an N such that for $n > N$, all the points $T_n^{-1}(R)$ lie in the region bounded by the arc $Q_1'Q_2'$ of U and the geodesic rays PQ_1' and PQ_2'. But then $T_n^{-1}(P_nR)$ is a segment of one of the hyperbolic rays with initial point P and terminal point in $Q_1'Q_2'$. This N fulfills the desired condition.

4. **On the number of transitive and intransitive geodesics when the hypothesis of unicity holds.** The hypothesis of unicity (Morse, 2, p. 56) is the assumption that given two points on U there is just one geodesic with these points as its points at infinity. By making use of known results on hyperbolic lines it is possible to extend the results of Morse (2, p. 63) concerning the number of transitive geodesic rays through a point of Ψ. The results concerning hyperbolic lines are due to Myrberg.

THEOREM OF MYRBERG. *Let F_M be a Fuchsian group of the first kind, with a finite set of generators and having a fundamental region either lying, together with its boundary, in Ψ or having all its vertices (necessarily parabolic) on U. There exists a set, T, of points of U with the property that every geodesic ray with a point of T as terminal point is transitive. The set T is of linear measure 2π. The set of points $I = U - T$ is non-denumerable and any geodesic ray with a point of I as terminal point is not transitive.*

The points of T will be called *transitive*, the points of I *intransitive*. These results concerning hyperbolic rays extend readily to the general case provided the hypothesis of unicity holds.

THEOREM 4.1. *If the hypothesis of unicity holds and F is a group of type F_M, every geodesic ray PQ_T, P in Ψ and Q_T in T, is transitive, while every geodesic ray PQ_I, P in Ψ and Q_I in I, is intransitive.*

For let (x, y, φ) be an arbitrary element in Ψ. From the hypothesis of unicity this element determines a unique directed geodesic, g_{AB}, with A as initial point at infinity and B as terminal point at infinity. Let P be any point of Ψ and Q_T any point of T. Since the hyperbolic ray PQ_T is transitive there exists a sequence, P_nQ_n, $n = 1, 2, \cdots$, of copies of PQ_T such that $\lim_{n\to\infty} P_n = A$ and $\lim_{n\to\infty} Q_n = B$, both in the euclidean sense. The sequence of directed geodesic

rays P_nQ_n has as unique limit geodesic the geodesic g_{AB} and the elements on the geodesic ray PQ_T must be everywhere dense among the totality of elements. Thus the geodesic ray PQ_T is transitive.

Conversely, if Q is a point of U, P is a point of Ψ and the geodesic ray PQ is transitive, then the hyperbolic ray PQ is transitive. Let (x, y, φ) again be an arbitrary element in Ψ but now let h_{AB} be the unique directed hyperbolic line determined by this element, the initial point at infinity being A, the terminal point B. Let g_{AB} be the unique directed geodesic determined by A and B and let (x', y', φ') be an element of g_{AB}. On the geodesic ray g_{PQ} with P as initial point and Q as point at infinity there must exist a sequence of elements (x_n, y_n, φ_n) with (x', y', φ') as limit element and such that if P_n is the point of g_{PQ} determining (x_n, y_n, φ_n), the geodesic distance PP_n becomes infinite with increasing n. But then there exist copies of g_{PQ} approximating closely arbitrarily great lengths of g_{AB} with (x', y') as midpoint. If P_i', $i = 1, 2, \cdots$, denote the initial points of these copies, and Q_i', $i = 1, 2, \cdots$, the terminal points, it follows that, in the euclidean sense, $\lim_{i\to\infty} P_i' = A$ and $\lim_{i\to\infty} Q_i' = B$. This implies that the hyperbolic ray is transitive. Thus no geodesic ray PQ_I can be transitive for this would imply that Q_I belonged to the set T.

BIBLIOGRAPHY

M. Morse (1), *A fundamental class of geodesics on any closed surface of genus greater than one*, Transactions of the American Mathematical Society, 26, 1924, pp. 25–61.

M. Morse (2), *Instability and transitivity*, Journal de Mathématiques pures et appliquées, Sér. 9, vol, 14, 1935, pp. 49–71.

P. J. Myrberg, *Ein Approximationssatz für die Fuchsschen Gruppen*, Acta Mathematica, 57, 1931, pp. 389–409.

O. Bolza, *Vorlesungen über Variationsrechnung*, 1909.

L. R. Ford, *Automorphic functions*, 1929.

P. Koebe, *Riemannsche Mannigfaltigkeiten und nichteuklidische Raumformen, Fünfte Mitteilung*, Sitzungsberichte der Preussischen Akademie der Wissenschaften, 1930, pp. 304–364.

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ON SUFFICIENT CONDITIONS IN THE PROBLEMS OF LAGRANGE AND BOLZA[1]

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1. **Introduction.** Sufficient conditions for a minimum in the problems of Lagrange and Bolza without assumptions of normality were first given by the author (V).[3] Three equivalent sets of sufficiency conditions were given. Except for trivial modifications and extensions these results were obtained in 1933 while the author was a Research Assistant to Professor Bliss at the University of Chicago. The author also gave a fourth set of sufficiency conditions involving the notion of conjugate points. These conditions were first established in the normal case by Bliss (IV, cf. II). For this last set the author assumed that the order of anormality of the arc under consideration was the same on every sub-interval x_1x_3. This assumption is not as restrictive as it seems since our problem can be transformed into one which has this property, at least if the values x_1 and x_2 determining the endpoints of the arcs are fixed. Moreover this normality assumption was used only to establish the existence of a conjugate system η_{ik}, ζ_{ik} of secondary extremals with a non-vanishing determinant $|\eta_{ik}|$ in case there is no conjugate point to the initial point of the arc under consideration. More recently Morse (VI) and Reid (VII) proved simultaneously and independently that this result could be obtained without the use of assumptions of normality. This result enabled them to prove with the help of the author's earlier results that the fourth set of sufficiency conditions referred to above is valid for the problem of Bolza without assumptions of normality. In the present paper we show that this fourth set can be obtained easily from the author's earlier results and that it is equivalent to sufficiency conditions already established by the author. Incidentally we also give, for the case in which only one end point is variable, sufficiency conditions involving the notion of focal points which are independent of assumptions of normality.

2. **Statement of the problem.** Let there be given a class of arcs

$$y_i = y_i(x) \qquad (x^1 \leqq x \leqq x^2; i = 1, \cdots, n) \tag{1}$$

and constants $(\alpha_1, \cdots, \alpha_r)$ satisfying the differential equations and end conditions

[1] Presented to the Society, December 28, 1934, under the title "A basic theorem in the problems of Lagrange and Bolza."

[2] The results here given were obtained while the author was a National Research Fellow 1933–34.

[3] Roman numerals in parentheses refer to the list of references at the end of this paper.

$$\begin{aligned} &\varphi_\beta(x, y, y') = 0 && (\beta = 1, \cdots, m < n), \\ &x^1 = x^1(\alpha_1, \cdots, \alpha_r), && y_i(x^1) = y_i^1(\alpha_1, \cdots, \alpha_r), \qquad (2) \\ &x^2 = a, && y_i(x^2) = b_i, \end{aligned}$$

where $(x, y) = (a, b)$ is a fixed point. We seek to find under what conditions a particular arc (1) and set (α) will afford a minimum to the functional

$$J = \theta(\alpha_1, \cdots, \alpha_r) + \int_{x^1}^{x^2} f(x, y, y')\, dx$$

in this class. This problem is called the *problem of Bolza with one variable end point*. In case $r = 0$ the initial point 1 as well as the final end point 2 is fixed. We obtain thereby the *problem of Lagrange* with fixed end points.

The hypotheses upon which our analysis is based, the notations and terminology used are those of the author (V, pp. 794–7). We denote the particular arc (1) and set (α) under consideration by the symbol g and make the following further assumptions concerning g. The set (α) belonging to g is the set $(\alpha) = (0)$. The matrix of the derivatives of the functions $x^1(\alpha)$, $y_i^1(\alpha)$ with respect to α_h has rank r at $(\alpha) = (0)$. Moreover the arc g is not tangent to the manifold (2) at its initial point 1. The functions $y_i(x)$ defining g are assumed to have continuous first and second derivatives and to satisfy the *Euler-Lagrange* equations

$$(d/dx)F_{y_i'} = F_{y_i}, \qquad \varphi_\beta = 0, \tag{3}$$

where $F = f + \lambda_\beta \varphi_\beta$,[4] with a set of multipliers $\lambda_\beta(x)$ which are continuous and have continuous derivatives. The arc g is accordingly an extremal arc. Moreover in case $r > 0$ we assume that the equation

$$(F - y_i' F_{y_i'})\, dx + F_{y_i'} dy_i - d\theta = 0 \tag{4}$$

is an identity in $d\alpha_h$ at the point 1 on g, when the differentials dx, dy_i, $d\theta$ are expressed in terms of $d\alpha_h$ with the help of equations (2). We shall assume further that g is *non-singular*, that is, that the determinant

$$\begin{vmatrix} F_{y_i' y_k'} & \varphi_{\gamma y_i'} \\ \varphi_{\beta y_k'} & 0 \end{vmatrix} \tag{5}$$

is different from zero on g.

The second variation of the functional J along g is expressible in the form (II, pp. 520-1)

$$J_2(\eta, w) = b_{hl} w_h w_l + \int_{x^1}^{x^2} 2\omega(x, \eta, \eta')\, dx \qquad (h, l = 1, \cdots, r),$$

[4] Here and elsewhere repeated subscripts denote summation.

where

$$2\omega = F_{y_i y_k}\eta_i\eta_k + 2F_{y_i y_k'}\eta_i\eta_k' + F_{y_i' y_k'}\eta_i'\eta_k' \qquad (i, k = 1, \cdots, n),$$

$$b_{hl} = \theta_{hl} - (F_x - y_i'F_{y_i})x_h^1x_l^1 - (F - y_i'F_{y_i'})x_{hl}^1$$
$$- F_{y_i}(x_h^1y_{il}^1 + x_l^1y_{ih}^1) - F_{y_i'}y_{ihl}^1.$$

The expression b_{hl} is evaluated at the point 1 on g. The symbols x^1, y_i^1 denote the functions $x^1(\alpha)$, $y_i^1(\alpha)$ and the subscripts h, l denote differentiation for α_h, α_l at $(\alpha) = (0)$. The matrix $\| b_{hl} \|$ is symmetric. The functions $\eta_i(x)$ are assumed to be continous on x^1x^2 and to possess continuous derivatives except possibly at a finite number of points on x^1x^2 and to satisfy with the constants w_h the equations

$$(6) \qquad \begin{aligned} &\Phi_\beta(x, \eta, \eta') = \varphi_{\beta y_i}\eta_i + \varphi_{\beta y_i'}\eta_i' = 0 \\ &\eta_i(x^1) - c_{ih}w_h = 0, \qquad \eta_i(x^2) = 0 \qquad (h = 1, \cdots, r) \end{aligned}$$

evaluated on g, where $c_{ih} = y_{ih}^1 - y_i'(x^1)x_h^1$. Such a set (η, w) will be called a *set of admissible variations*. The matrix $\| c_{ih} \|$ has rank r because of the non-tangency condition on g (II, pp. 525-6). It is clear that in case $r = 0$ the constants b_{hl}, c_{ih}, w_h do not appear.

The analogues of the equations (3) and (4) for the functional $J_2(\eta, w)$ subject to the conditions (6) are the equations

$$(7) \qquad (d/dx)\,\Omega_{\eta_i'} = \Omega_{\eta_i}, \qquad \Phi_\beta = 0,$$

$$(8) \qquad \zeta_i(x^1)c_{ih} - b_{hl}w_l = 0 \qquad (h, l = 1, \cdots, r),$$

where $\Omega = \omega + \mu_\beta\Phi_\beta$, $\zeta_i = \Omega_{\eta_i'}$. By a *secondary extremal* will be meant a set of functions $\eta_i(x)$, $\mu_\beta(x)$ having continuous derivatives η_i', η_i'', μ_β' and satisfying equations (7). Since the determinant (5) is different from zero along g a secondary extremal η_i, μ_β is uniquely determined by the values of η_i, $\zeta_i = \Omega_{\eta_i'}$ at a single point $x = x^0$ (I, p. 725). In the sequel it will be convenient to denote a secondary extremal by the symbols η_i, ζ_i as well as by η_i, μ_β.

A value $x^3 \neq x^1$ will be said to define a *focal point* 3 *of the manifold* (2) on g if there exists a secondary extremal η_i, ζ_i satisfying the equations

$$(9) \qquad \eta_i(x^1) - c_{ih}w_h = 0, \qquad \zeta_i(x^1)c_{ih} - b_{hl}w_l = 0$$

with a set of constants w_h, having $\eta_i(x^3) = 0$, and having $(\eta) \not\equiv (0)$ on x^1x^3. In case $r = 0$ the conditions (9) reduce to $\eta_i(x^1) = 0$, and focal points are called *conjugate points of the point* 1 on g. One can prove in the usual manner that under suitable normality assumptions there can be no focal point 3 on g between 1 and 2 if g is to be a minimizing arc (cf V, pp. 802-3).

The extremal g will be said to satisfy the *Weierstrass condition* II_N' if at each element (x, y, y', λ) in a neighborhood N of those on g the inequality

$$E(x, y, y', \lambda, Y') > 0$$

holds for every admissible set $(x, y, Y') \neq (x, y, y')$, where $E(x, y, y', \lambda, Y')$ is the Weierstrass E-function (I, p. 718). The arc g will be said to satisfy the *Clebsch condition* III′ if at each element (x, y, y', λ) on g the inequality

$$F_{y_i' y_k'} \pi_i \pi_k > 0$$

holds for every solution $(\pi) \neq (0)$ of the equations $\varphi_{\beta y_i'} \pi_i = 0$. It is well known that the condition III′ for g implies that g is non-singular (I, p. 735).

It is the purpose of this paper to establish the following theorems:

THEOREM 1. *If the extremal g satisfies the transversality condition* (4), *the Weierstrass condition* II_N', *the Clebsch condition* III′, *and has on it no focal point* 3 *of the manifold* (2) *then g affords a proper strong minimum to the functional J relative to neighboring admissible arcs satisfying the conditions* (2).

In case $r = 0$ Theorem 1 can be stated more simply as follows:

THEOREM 2. *If an extremal g for the problem of Lagrange with fixed end points satisfies the conditions* II_N', III′, *and has on it no point* 3 *conjugate to* 1 *then g affords a proper strong minimum to J relative to neighboring admissible arcs joining its end points.*

Turning to the general problem of Bolza as discussed by the author we obtain the further result:

THEOREM 3. *Theorems* 8:3 *and* 9:4 *of Hestenes* (V, pp. 811, 816) *hold true without assumptions of normality.*

3. **Proofs of Theorems** 1, 2, **and** 3. The proofs of Theorems 1, 2, and 3 are based on a sequence of eight lemmas. We begin with the following one:

LEMMA 1. *Let $\bar\eta_{i\gamma}, \bar\zeta_{i\gamma}$ $(\gamma = 1, \cdots, p)$ be a maximal set of linearly independent secondary extremals satisfying the equations* (9) *with $w_h = 0$ and having $\bar\eta_{i\gamma} \equiv 0$ on x^1x^2. If $\eta_i(x)$ is a solution of the equations $\Phi_\beta = 0$ and satisfies the equations*

$$\eta_i(x^1) = c_{ih} w_h + \bar\zeta_{i\gamma}(x^1) w_{r+\gamma}, \qquad \eta_i(x^2) = 0 \qquad (h = 1, \cdots, r) \tag{10}$$

with a set of constants $w_1, \cdots, w_{r+p}$, then the constants $w_{r+1}, \cdots, w_{r+p}$ are all zero, that is, $\eta_i(x)$ satisfies the equations (6) *with the set of constants $w_1, \cdots, w_r$.*

To prove this let $\mu_{\beta\gamma}(x)$ be the multipliers belonging to the secondary extremals $\bar\eta_{i\gamma}, \bar\zeta_{i\gamma}$. For these extremals the equations (7) take the form

$$(d/dx)\, \mu_{\beta\gamma} \varphi_{\beta y_i'} = \mu_{\beta\gamma} \varphi_{\beta y_i}. \tag{11}$$

Multiplying the equations $\Phi_\beta = 0$ by $\mu_{\beta\gamma}$, adding, and integrating from x^1 to x^2, it is found by the use of equations (11) and (10) that

$$0 = \mu_{\beta\gamma}\varphi_{\beta y_i'} \eta_i \Big|_1^2 = \bar\zeta_{i\gamma}\eta_i \Big|_1^2 = -\bar\zeta_{i\gamma}(x^1) c_{ih} w_h - \bar\zeta_{i\gamma}(x^1)\bar\zeta_{i\tau}(x^1) w_{r+\tau} \qquad (\gamma, \tau = 1, \cdots, p).$$

The first member on the right is zero since $\bar\eta_{i\gamma}, \bar\zeta_{i\gamma}$ satisfies equations (9) with $w_h = 0$. The determinant $| \bar\zeta_{i\gamma}(x^1)\bar\zeta_{i\tau}(x^1) |$ is different from zero since the secondary extremals $\bar\eta_{i\gamma}, \bar\zeta_{i\gamma}$ are linearly independent. It follows that the constants $w_{r+1}, \cdots, w_{r+p}$ are all zero, as was to be proved.

We shall now define conditions IV and IV′. Let $\bar\eta_{i\gamma}, \bar\zeta_{i\gamma}$ be a set of secondary extremals having the properties described in Lemma 1, and let η_{ik}, ζ_{ik} be a set of n linearly independent secondary extremals satisfying the conditions

$$
(12) \qquad \begin{aligned} \eta_i(x^1) &= c_{ih}w_h + \bar\zeta_{i\gamma}(x^1)w_{r+\gamma} & (h, l = 1, \cdots r; \gamma = 1, \cdots, p) \\ \zeta_i(x^1)c_{ih} - b_{hl}w_l &= 0\,, \qquad \zeta_i(x^1)\bar\zeta_{i\gamma}(x^1) = 0 & \end{aligned}
$$

with a set of constants w_{qk} $(q = 1, \cdots, r + p)$. Let u_{ik}, v_{ik} be a set of n linearly independent secondary extremals having $u_{ik}(x^2) = 0$. The *condition* IV will be said to hold on x^1x^2 if at each point x on x^1x^2 the inequality

$$
(13) \qquad (\zeta_{ij}u_{ik} - \eta_{ij}v_{ik})a_jb_k \geqq 0 \qquad (i, j, k = 1, \cdots, n)
$$

holds for every set of constants (a_j, b_k) satisfying the equations

$$
(14) \qquad \eta_{ij}(x)a_j = u_{ik}(x)b_k.
$$

By the *condition* IV′ on x^1x^2 will be meant the condition IV on x^1x^2 with the added assumption that the determinant

$$
(15) \qquad |\, \zeta_{ij}u_{ik} - \eta_{ij}v_{ik} \,|
$$

is different from zero. The condition IV′ of the present paper is essentially a special case of the "condition IV′" defined by Hestenes (V, p. 806) in a more general situation, although modified somewhat so as to simplify our proofs. A similar simplification can be made in the more general case.

It is well known that the expression $\zeta_iu_i - \eta_iv_i$ formed for two secondary extremals η_i, ζ_i and u_i, v_i is a constant (I, p. 738). If this constant is zero the secondary extremals are said to be conjugate to each other. A set of n mutually conjugate linearly independent secondary extremals is said to form a *conjugate system.* The systems η_{ik}, ζ_{ik} and u_{ik}, v_{ik} appearing in the definitions of the conditions IV and IV′ are conjugate systems, as follows readily by an examination of their values at x^1 and x^2 respectively and by the use of equations (12). It is clear also that the elements of the determinant (15) are all constants.

When the determinant (15) is different from zero the condition IV can be simplified somewhat. This simplification is described in the following lemma:

LEMMA 2. *If the determinant* (15) *is different from zero then the secondary extremals* η_{ik}, ζ_{ik} *and* u_{ik}, v_{ik} *described above can be chosen so that*

$$
(16) \qquad \zeta_{ij}u_{ik} - \eta_{ij}v_{ik} = \delta_{jk} \qquad [\delta_{jj} = 1,\ \delta_{jk} = 0\ (j \neq k)].
$$

The condition IV *then holds on* x^1x^2 *if and only if the inequality*

$$
(17) \qquad \eta_{ij}u_{kj}c_ic_k \geqq 0 \qquad (i, j, k = 1, \cdots, n)
$$

holds on x^1x^2 *for every set of constants* $c_1, \cdots, c_n$.

The equations (16) follow at once if we replace the system u_{ik}, v_{ik} by the system $u_{ij}d_{jk}, v_{ij}d_{jk}$, where the determinant $|\, d_{jk} \,|$ is the inverse of the determinant (15). To prove the last statement of the lemma let C, D be the matrices

$$C = \begin{Vmatrix} \eta_{ik} & u_{ik} \\ \zeta_{ik} & v_{ik} \end{Vmatrix}, \qquad D = \begin{Vmatrix} -v_{ik} & \zeta_{ik} \\ u_{ik} & -\eta_{ik} \end{Vmatrix}.$$

If the transpose of D is denoted by D' and the identity matrix by I then the matrix equation $D'C = I$ holds by virtue of equations (16) and the conjugacy of the systems η_{ik}, ζ_{ik} and u_{ik}, v_{ik}. We have accordingly $CD' = I$ and from this equation it is found that $\eta_{ij}u_{kj} = u_{ij}\eta_{kj}$. It follows that the set $a_{jk} = u_{kj}$, $b_{jk} = \eta_{kj}$ forms a maximal set of solutions of equations (14) and hence that the condition (13) subject to (14) is equivalent to the condition (17), as was to be proved.

LEMMA 3. *The determinant* (15) *is zero if and only if the final end point* 2 *on* g *is a focal point of the manifold* (2).

For if the determinant (15) is evaluated at $x = x^2$ it is found that this determinant vanishes if and only if the determinant $| \eta_{ik}(x^2) |$ is zero. If the latter determinant vanishes there exist constants a_j not all zero such that $\eta_{ij}a_j = 0$ at $x = x^2$. The secondary extremal $\eta_i = \eta_{ij}a_j$, $\zeta_i = \zeta_{ij}a_j$ will have $\eta_i(x^2) = 0$ and will satisfy equations (12) with $w_q = w_{qj}a_j$ ($q = 1, \cdots, r + p$), and hence also equations (9) with the constants $w_1, \cdots, w_r$, by Lemma 1. Moreover $(\eta) \not\equiv (0)$ since otherwise η_i, ζ_i would be linearly dependent on the secondary extremals $\bar{\eta}_{i\gamma}$, $\bar{\zeta}_{i\gamma}$ defined in Lemma 1 and hence could not satisfy the equations (12) unless the a's were all zero which is not the case. It follows that if $| \eta_{ik}(x^2) | = 0$ the point 2 is a focal point of the manifold (2). Conversely if there exists a secondary extremal η_i, ζ_i having $(\eta) \not\equiv (0)$, vanishing at x^2, and satisfying the conditions (9) with a set of constants w_h, such an extremal will be linearly dependent on the secondary extremals η_{ik}, ζ_{ik} and $\bar{\eta}_{i\gamma}$, $\bar{\zeta}_{i\gamma}$. Consequently in this case the determinant $| \eta_{ik}(x^2) |$ must be zero since $\bar{\eta}_{i\gamma} \equiv 0$ on x^1x^2. This proves Lemma 3.

LEMMA 4. *If* $J_2(\eta, w) > 0$ *for every admissible variation* $(\eta, w) \not\equiv (0, 0)$, *then the condition* IV′ *holds on* x^1x^2.

To prove this let a_k, b_k be a set of constants satisfying equations (14) at a value $x = x^3$ and let η_i, ζ_i be the broken secondary extremal defined by the equations

$$\eta_i = \eta_{ik}a_k, \qquad \zeta_i = \zeta_{ik}a_k \qquad (x^1 \leqq x \leqq x^3),$$
$$\eta_i = u_{ik}b_k, \qquad \zeta_i = v_{ik}b_k \qquad (x^3 \leqq x \leqq x^2).$$

The broken extremal η_i, ζ_i satisfies equations (10) with $w_q = w_{qk}a_k$ ($q = 1, \cdots, r + p$) and hence also equations (6) with the set $w_1, \cdots, w_r$, by Lemma 1. If we denote by μ_β the multipliers belonging to η_i, ζ_i and note that

$$2\,\Omega = \eta_i\Omega_{\eta_i} + \eta_i'\,\Omega_{\eta_i'} + \mu_\beta\Omega_{\mu_\beta}$$

then by the use of equations (9) and (14) it is found that along η_i, ζ_i, w_k we have

$$J_2(\eta_1 w) = b_{hl}w_hw_l + \int_{x^1}^{x^3} 2\,\Omega\,dx + \int_{x^3}^{x^2} 2\,\Omega\,dx$$

$$= b_{hl}w_hw_l + [\eta_i\zeta_i]_1^2 + [\eta_i\zeta_i]_{x^3+0}^{x^3-0}$$

$$= \eta_i(x^3+0)\,\zeta_i(x^3-0) - \eta_i(x^3-0)\,\zeta_i(x^3+0)$$

$$= (\zeta_{ij}u_{ik} - \eta_{ij}v_{ik})\,a_jb_k\,.$$

Clearly this expression must be positive or zero. Hence the condition IV holds. The condition IV′ also holds. For otherwise the point 2 would be a focal point of the manifold (2), by Lemma 3, and there would exist a secondary extremal η_i, ζ_i having $(\eta) \not\equiv (0)$ and $\eta_i(x^2) = 0$, and satisfying equations (9) with a set of constants w_h. Moreover we would have $J_2(\eta, w) = 0$ for this extremal, as can be seen by an argument like that given above. This proves Lemma 4.

LEMMA 5. *If the condition* IV′ *holds on* x^1x^2, *then there exists a conjugate system* U_{ik}, V_{ik} *of secondary extremals having* $|U_{ik}(x)| \neq 0$ *on* x^1x^2 *and such that if* $r > 0$ *the inequality*

$$b_{hl}z_hz_l - U_{ij}(x^1)V_{ik}(x^1)a_ja_k > 0 \tag{18}$$

holds for every set of constants $(a_k, z_h) \neq (0, 0)$ *satisfying the equations*

$$U_{ik}(x^1)a_k - c_{ih}z_h = 0\,. \tag{19}$$

For, suppose equations (16) hold. Then the secondary extremals $U_{ik} = \eta_{ik} + u_{ik}$, $V_{ik} = \zeta_{ik} + v_{ik}$ have the properties described in the lemma. That they form a conjugate system follows readily from equations (16) and the conjugacy of the systems η_{ik}, ζ_{ik} and u_{ik}, v_{ik}. Moreover if the determinant $|U_{ik}|$ were zero at a value x^3 on x^1x^2 then there would exist constants $c_1, \cdots, c_n$ not all zero such that at $x = x^3$ we would have

$$U_{ik}c_k = \eta_{ik}c_k + u_{ik}c_k = 0\,.$$

The equations (14) would then be satisfied at $x = x^3$ by the set $a_k = c_k$, $b_k = -c_k$ and for these constants the bilinear form (13) would take the value $-c_kc_k < 0$, contrary to the condition IV′. The last part of the lemma can be proved by the same methods as those used by the author in a similar situation (V, pp. 807-9).

LEMMA 6. *If the Clebsch condition* III′ *holds and if there is a conjugate system* U_{ik}, V_{ik} *of secondary extremals having the properties described in Lemma 5, then* $J_2(\eta, w) > 0$ *for every set of admissible variations* $(\eta, w) \neq (0, 0)$. *Moreover the condition* IV′ *holds.*

The first part of the theorem is well known for the case $r = 0$. In the more general case considered here the proof is precisely like that of "Theorem 8:1" of Hestenes (V, p. 810). The second part of the theorem follows from Lemma 4.

LEMMA 7. *If the condition* IV *holds on* x^1x^2 *for all values of* x^2 *on* $x^1 < x^2 < x^3$ *and if the value* x^3 *does not define a focal point of the manifold* (2) *then the condition* IV′ *holds for* $x^2 = x^3$.

For, let $\bar\eta_{i\gamma}$, $\bar\zeta_{i\gamma}$ be a maximal set of linearly independent secondary extremals having $\bar\eta_{i\gamma} \equiv 0$ on x^1x^3 and satisfying the equations (9) with $w_h = 0$ and let η_{ik}, ζ_{ik} be a conjugate system satisfying the corresponding equations (12) with a

set of constants w_{qk}. Let $u_{ik}(x, x^2)$, $v_{ik}(x, x^2)$ be a conjugate system of secondary extremals varying continuously with x^2 and having $u_{ik}(x^2, x^2) = 0$. Since the value x^3 does not define a focal point of the manifold (2) the determinant (15) formed for these functions is different from zero for $x^2 = x^3$ and hence also for $x^2 < x^3$ and sufficiently near x^3. We may accordingly suppose that the equations (16) hold for these values of x^2. It follows from the condition IV that the quadratic form (17) formed for these functions is positive on x^1x^2 for $x^2 < x^3$ and sufficiently near x^3 and hence also for $x^2 = x^3$, since its coefficients are continuous in x^2. The condition IV′ accordingly holds for $x^2 = x^3$, as was to be proved.

LEMMA 8. *If the Clebsch condition* III′ *holds and there is no focal point of the manifold* (2) *on* g, *then the condition* IV′ *holds on* x^1x^2.

To prove this we note that by virtue of Lemma 6 the condition IV′ surely holds if x^2 is sufficiently near x^1 since the conjugate system U_{ik}, V_{ik} with $|U_{ik}| \neq 0$ at $x = x^1$ and $U_{ih}(x^1) = c_{ih}$, $V_{ih}(x^1) = -\sigma c_{ih}$ $(h = 1, \cdots, r)$ has the properties described in Lemma 5, at least if σ is chosen sufficiently large. If the condition IV′ did not hold for our original x^2 let x^3 be the least upper bound of the values of x^2 for which this condition holds. Since x^3 does not define a focal point of the manifold (2) the condition IV′ holds also for $x^2 = x^3$, by Lemma 7. The conjugate system U_{ik}, V_{ik} described in Lemma 5 would then have $|U_{ik}| \neq 0$ on x^1x^2 for x^2 slightly larger than x^3 and the condition IV′ would hold for these values of x^2, by Lemma 6. This contradicts our choice of x^3 and the lemma is proved.

We are now in position to prove Theorems 1, 2, and 3. In order to prove Theorem 1 we use a result established by the author in an earlier paper (V, p. 815), namely, that the extremal g affords a proper strong relative minimum to the functional J if it satisfies the conditions (4), II_N', III′, and the inequality $J_2(\eta, w) > 0$ holds for every set of admissible variations $(\eta, w) \neq (0, 0)$. With this in mind Theorem 1 follows at once by the use of Lemmas 8, 5, and 6.

Theorem 2 is an obvious corollary of Theorem 1. The theorem however can be proved without reference to the problem of Bolza. For, Lemmas 8 and 5 tell us that there exists a conjugate system U_{ik}, V_{ik} of secondary extremals having $|U_{ik}| \neq 0$ on x^1x^2. The proof of this result given above does not depend in this special case on the references made to the problems of Bolza. The extremal g is accordingly an extremal of a Mayer field (III, p. 576-7). The theorem now follows from the usual fundamental sufficiency theorem (I, pp. 731-2) with the help of the condition II_N'.

In order to prove Theorem 3 we note that in the proofs of "Theorems 8:3 and 9:4" of Hestenes (V, pp. 811, 816) the assumptions regarding normality were made in order to insure the existence of a conjugate system U_{ik}, V_{ik} having $|U_{ik}| \neq 0$ on x^1x^2. In view of the above remarks it is clear that these assumptions of normality are not needed, and Theorem 3 is established.

REFERENCES

I. Bliss, *The problem of Lagrange in the calculus of variations*, American Journal of Mathematics, vol. 52 (1930), pp. 673–744.

II. Morse, *Sufficient conditions in the problem of Lagrange with variable end points*, American Journal of Mathematics, vol. 53 (1931), pp. 517–596.

III. Morse, *Sufficient conditions in the problem of Lagrange with fixed end points*, Annals of Mathematics, (2), vol. 32 (1931), pp. 567–577.

IV. Bliss, *The problem of Bolza in the calculus of variations*, Annals of Mathematics, (2), vol. 33 (1932), pp. 261–274.

V. Hestenes, *Sufficient conditions for the problem of Bolza in the calculus of variations*, Transactions of the American Mathematical Society, vol. 36 (1934), pp. 793–818.

VI. Morse, *Sufficient conditions in the problem of Lagrange without assumptions of normalcy*, Transactions of the American Mathematical Society, vol. 37 (1935), pp. 147–160.

VII. Reid, *The theory of the second variation for the non-parametric problem of Bolza*. American Journal of Mathematics, vol. 57 (1935), pp. 573–585.

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ON THE SINGULAR SOLUTIONS OF ALGEBRAIC DIFFERENTIAL EQUATIONS

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The singular solutions of algebraic differential equations of the first order were exhaustively treated by Hamburger.[1] For equations of higher order, aside from an interesting first result of Hamburger,[2] derived by the same method which he used for the first order, very little of a systematic character has been published.[3]

The present paper studies singular solutions from the algebraic point of view developed in our book *Differential Equations From The Algebraic Standpoint*.[4] We deal with an algebraically irreducible differential polynomial[5] F in the unknown function y of the independent variable x. The coefficients in F are understood to belong to a given field of meromorphic functions of x. If we represent the j^{th} derivative of y by y_j, the equation $F = 0$ may be written

$$F(x; y, y_1, \cdots, y_n) = 0, \tag{a}$$

where it is understood that $n \geqq 1$. By a singular solution of (a), we understand here any solution of (a) which annuls the *separant* of F, that is, $\partial F/\partial y_n$.

The manifold of F is composed of a finite number of irreducible manifolds, one of which is the *general solution* of F. The general solution contains all of the non-singular (normal[6]) solutions of F, and sometimes, in addition, singular solutions. If there are irreducible manifolds distinct from the general solution, they are made up of singular solutions.

The problem which we consider here, and which we solve completely for equations of the second order, is that of determining the distribution of the singular solutions of F among the irreducible manifolds in the manifold of F.[7] It is a

[1] Mathematische Annalen, vol. 112, (1893), p. 205.

[2] Ibid., vol. 121, (1899), p. 265 and vol. 122, (1900), p. 322.

[3] See Mayer, Mathematische Annalen, vol. 22, (1883), p. 368; Goursat, American Journal, vol. 11, (1889), p. 329; Fine, American Journal, vol. 12, (1889), p. 295; Hudson, Proc. London Math. Soc., vol. 33, (1901), p. 380; Burgatti, Palermo Rendiconti, vol. 20, (1905), p. 256; Cerf, Journal de Mathématiques, 9th series, vol. 8, (1929), p. 161. The discussion, based on work of Dixon, which is to be found in Forsythe's *Differential Equations*, vol. 2, pp. 247-275, is interesting only to the extent that it describes what happens in certain examples. As a piece of general theory, it is unsound.

[4] Colloquium Publications of the American Mathematical Society, vol. XIV, New York, 1932. Designated below by *A. D. E.*

[5] Differential *polynomial* stands here for differential *form* as in *A. D. E.* We continue to use *form* as an abbreviation.

[6] *A. D. E.*, §72.

[7] For equations of the first order, this determination was made, essentially, in *A. D. E.*, §89. Cf. remarks below relative to Hamburger's work.

question of describing the composition of each of the irreducible manifolds and also one of determining those irreducible manifolds to which a given singular solution of F belongs.

Under the second phase of the problem, one might, for instance, for a given singular solution of F, ask whether it is contained in the general solution of F. For the case of $n = 2$, we shall show how that can be decided.

Our problem is thus one of making a first separation of singular solutions into types. This separation, which is effected by assigning each singular solution to one or more irreducible manifolds, does not exhaust the theory of singular solutions. The classes of singular solutions with which our present work deals require further study, to bring out the properties of the singular solutions in greater detail. In particular, it seems desirable to study the possibilities for a singular solution as an envelope of non-singular solutions or of other singular solutions. Singular solutions are very often such envelopes. The examples in §91 will give some indication of what may be expected here.

We shall now describe, to some extent, our specific results.

The manifold of F consists of the general solutions of a certain number of forms. In Chapter V of *A. D. E.*, it is shown how to find a set of forms whose general solutions make up the manifold of F. However, the general solutions of the forms there obtained need not all be essential manifolds. There arises thus the following problem. *Let A be an algebraically irreducible form whose general solution consists of singular solutions of F. It is required to determine whether the general solution of A is an essential manifold in the manifold of F.*

A solution of this problem, based on a finite number of rational operations and differentiations, is given in the first part of the paper. We are thus able to produce a set of forms whose general solutions are the essential irreducible manifolds in the manifold of F.

The interesting special case of $A = y$ admits of simple description. The answer to the problem is, in that case, as follows. *Let F vanish for $y = 0$. For $y = 0$ to be an essential manifold, it is necessary and sufficient that F, considered as a polynomial in $y, y_1, \cdots, y_n$, contain a term in y alone, that is, a term free of $y_1, \cdots, y_n$, which is of lower degree than every other term in F.*

For instance, $y = 0$ is an essential manifold for $y_1y_2 - y$ but not for $yy_3 - y_2$ or for $y_2y_3 - y^2$.

For the special case in which A is of order $n - 1$ in y, the above problem is closely related to Hamburger's paper on equations of higher order.[8] Hamburger derives conditions for the solution of $A = 0$, with A of order $n - 1$, to be a "singular integral" or a "particular integral" of $F = 0$. These conditions, when formulated algebraically, become our conditions for the general solution of A to be essential or unessential. Hamburger did not possess the concepts of irreducible manifold and of general solution, which throw new light on his special result.[9]

[8] Mathematische Annalen, vol. 121.

[9] While Hamburger's method is suited only for the case in which A is of order $n - 1$, it gives some interesting information, in that case, as to the nature of the solutions of A as envelopes of non-singular solutions of F.

In proving the sufficiency of our condition for the general solution of A to be essential, we use a transformation belonging to the class of transformations which was discovered by Painlevé and applied by him to the study of differential equations whose solutions have fixed critical points.[10] This would suggest a search for analogies between the critical point problem and the problem of singular solutions. One might note that Hamburger's methods were suggested by those which Fuchs employed for the study of the critical points of the solutions of equations of the first order. In any case, it is interesting that Painlevé's transformation, which presented itself spontaneously in our work, and which operates here in a rather new manner, should have found another application.

The second, and longer, part of the paper, deals with equations of the second order. The problem of the distribution of the singular solutions is formulated in detail in §§19, 20. The following special question is found to hold the chief rôle. *Let F, an algebraically irreducible form in y, of order 2, vanish for $y = 0$. It is required to determine whether the solution $y = 0$ belongs to the general solution of F.*

In the discussion of this question, series of the type

$$\text{(b)} \qquad \varphi_1(x)y^{q/r} + \cdots + \varphi_k(x)y^{(q+k)/r} + \cdots$$

where q and r are positive integers with $q \geqq r$ and where the φ are analytic functions of x, are found to be important. The series (b) is so constructed that the purely formal differential equation

$$\text{(c)} \qquad y_1 = \varphi_1(x)y^{q/r} + \cdots + \varphi_k(x)y^{(q+k)/r} + \cdots$$

formally implies $F = 0$. When $y = 0$ is in the general solution of F, a series (b) exists such that if G is any form which vanishes for every solution in the general solution of F, (c) formally implies $G = 0$.

Only the formal properties of the series (b) are important, and there are actually cases in which the series diverges for every $y \neq 0$. Convergent series (b) appear in Hamburger's work on equations of the first order. For our purposes, it is necessary to develop a fairly general theory of the series (b) whose associated differential equations (c) imply, for a form F of the second order, $F = 0$.[11]

The method of testing for the presence of $y = 0$ in the general solution of F is summarized in §87.

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[10] Bulletin de la Société Mathématique de France, vol. 28, (1900), p. 201.

[11] When F is of the first order, it is a matter of solving the equation $F = 0$ considered as an algebraic equation for y_1. When F is of the first order, the presence of $y = 0$ in the general solution of F depends only on the exponents in F. For the second order, the question depends on the coefficients as well. For instance, $y = 0$ belongs to the general solution of $(y_2 - y_1)^2 - (y_1 + y)$, but not to that of $(y_2 - y_1)^2 - (y_1 - y)$. The difference between the cases of first order and second order can be judged from the fact that the first order problem is settled in a single section of *A. D. E.* (§89).

Part I. Essential Manifolds

Preparation Process

1. We consider forms in a single unknown y. By the *order* of such a form, we shall mean the order of the form in y.

Let F and A be two forms, effectively involving y and algebraically irreducible,[12] of the respective orders m and $n < m$. Let A_j represent the j^{th} derivative of A, and S the separant of A. We shall show *the existence of a non-negative integer t and of a positive integer r such that S^tF has a representation*

$$(1) \qquad \sum_{j=1}^{r} C_j A^{p_j} A_1^{i_{1j}} A_2^{i_{2j}} \cdots A_{m-n}^{i_{m-n,j}}$$

with non-negative p_j and i_{kj}, where no two of the r sets $i_{1j}, \cdots, i_{m-n,j}$ are identical;[13] *the C_j being forms of orders not exceeding n which are not divisible by A.*[14]

We start with the case of $m = n + 1$. Let F be of degree a in y_{n+1}. Then S^aF can be written as a polynomial in Sy_{n+1} with coefficients of orders not exceeding n. Now

$$A_1 = Sy_{n+1} + T,$$

with T of order n at most. Thus S^aF can be written as a polynomial in $A_1 - T$, and hence as a polynomial in A_1, with coefficients of orders n at most. If we write each coefficient in the form CA^p, $(p \geqq 0)$, with C not divisible by A, we have a representation (1) for S^aF.

Suppose now that (1) can be produced for $m < s$, where $s > n + 1$. We make an induction to $m = s$. Let F, of order s, be of degree a in y_s. We see as above that S^aF can be written as a polynomial in A_{s-n} with coefficients of orders less than s. For a sufficiently large positive integer b, the product of any of these coefficients by S^b will have a representation (1). Hence, for t large, S^tF can be written as a polynomial in $A_1, \cdots, A_{s-n}$ with coefficients of orders not exceeding n. Treating these coefficients as above, we have (1) for S^tF.

2. We shall now show that, for any admissible t, (1) is unique.[15] Let S^tF have two representations (1). If $y, y_1, \cdots, y_n$ are taken arbitrarily as analytic functions of x, with the sole restriction that S does not vanish, then $y_{n+1}, \cdots, y_m$ can be determined in succession so as to make $A_1, \cdots, A_{m-n}$ become any desired analytic functions. This shows that the same power products in $A_1, \cdots, A_{m-n}$ appear in both representations and that the respective coefficients are identical as polynomials in $y, y_1, \cdots, y_n$. The uniqueness of (1) follows immediately.

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[12] The irreducibility of F and A is used only in the uniqueness discussions, and in the later applications.

[13] The omission of p_j will enable us later to choose a unique representation (1) in a simple way.

[14] This implies that no C_j is zero.

[15] The irreducibility of F and A is not necessary here.

We see now, because S is not divisible by A, that, for two distinct values t_1 and t_2 of t, with $t_2 > t_1$, (1) is the same except that the C_j for t_2 are those for t_1 multiplied by $S^{t_2-t_1}$.[16]

3. By taking t as small as possible, we are led to a unique expression (1). In all our later work it will be understood that the smallest admissible t is used.

In actual calculation the smallest t can be found as follows. If S is free of y we take $t = 0$. If S involves y, we first secure (1) with any admissible t and then determine the highest power S^q of S which is a factor of every C_j. As F is algebraically irreducible, S^t must be divisible by S^q. A division by S^q will thus give the unique representation sought. In this expression, any common factor of the C_j will be a factor of S.

The Condition

4. If a system of forms, Σ_1, holds a system Σ_2, we shall say that Σ_1 *holds* the *manifold* of Σ_2.

Suppose that F holds the general solution of A. Then there is no term in (1) free of A and its derivatives. Otherwise some C_j would hold the general solution of A. This is impossible, since the C_j are of order n at most and not divisible by A.

5. Let F hold the general solution of A. We are going to prove that *for the general solution of A to be an essential irreducible manifold in the manifold of F, it is necessary and sufficient that* (1) *possess a term of the type $C_jA^{p_j}$, which term, if* (1) *is considered as a polynomial in $A, A_1, \cdots, A_{m-n}$, is of lower degree than every other term in* (1).[17]

Sufficiency Proof

6. Let the condition be satisfied. We shall prove that the general solution of A is an essential manifold.

Suppose that this is not so. Then the general solution of A is a proper part of some essential irreducible manifold in the manifold of F. This essential manifold will be the general solution of a form B whose order exceeds that of A. By *A. D. E.*, §74, given any solution $\bar{y}$ in the general solution of A, there is a set of points, dense in the area in which $\bar{y}$ is analytic, such that, given any point a of the set, any positive integer μ and any $\epsilon > 0$, we can find a solution $\tilde{y}$ in the general solution of B which is analytic at a and does not annul A at a,[18] with

$$|\tilde{y}_j(a) - \bar{y}_j(a)| < \epsilon, \qquad j = 0, 1, \cdots, \mu.$$

[16] We are now using the irreducibility of A.

[17] That is, for $k \neq j$, $p_j < p_k + i_{1k} + \cdots + i_{m-n,\,k}$. If a term $C_jA^{p_j}$ exists, (1) must contain other terms, since $m > n$.

[18] In the statement of this theorem in *A. D. E.*, it is not emphasized that $\tilde{y}$ does not annul A at the special point a. This fact, however, stands out in the proof. Cf. *A. D. E.*, p. 102, line 27.

7. Let $C_1A^{p_1}$ be the term of lowest degree in (1).

In choosing a solution $\bar{y}$ as in §6, we take $\bar{y}$ so as not to annul C_1. We take a so that the coefficients in A and the C_j are analytic at a and so that C_1 does not vanish at a for the particular $\bar{y}$ with which we are dealing.

Consider, for $j = 2, \cdots, r$, the fractions, with positive denominators and positive values,

$$\frac{\sum_{k=1}^{m-n} k i_{kj}}{p_j - p_1 + \sum_{k=1}^{m-n} i_{kj}}. \tag{2}$$

Let s be any number which exceeds each of these fractions.

We denote by c a positive real constant which will later be made small.[19] Let

$$z = \frac{x - a}{c}. \tag{3}$$

For any $\tilde{y}$ approximating to $\bar{y}$ as in §6, A becomes a function of x, analytic at a and distinct from zero at a. Let

$$w(z) = c^{-s}A \tag{4}$$

where $\tilde{y}$ is understood to be substituted into A[20] and where the x used in A is related to z as in (3). Then w is analytic at $z = 0$ and

$$w_1 = c^{-s+1}A_1, \cdots, w_{m-n} = c^{-s+m-n}A_{m-n}$$

where the subscripts of w indicate differentiation with respect to z. Let

$$D_j(z) = C_j$$

with $\tilde{y}$ substituted into C_j.

The j^{th} term in (1) goes over into

$$c^{u_j}\, D_j(z)\; w^{p_j}\, w_1^{i_{1j}} \cdots w_{m-n}^{i_{m-n,j}} \tag{5}$$

where

$$u_j = s\left(p_j + \sum_{k=1}^{m-n} i_{kj}\right) - \sum_{k=1}^{m-n} k i_{kj}.$$

We have $u_1 = p_1 s$ and, referring to (2), we see that $u_j > p_1 s$ for $j > 1$.

As (1) vanishes for $y = \tilde{y}$, we can write

$$D_1(z)\, w^{p_1} + \sum_{j=2}^{r} c^{u_j - p_1 s}\, D_j(z)\, w^{p_j} \cdots w_{m-n}^{i_{m-n,j}} = 0. \tag{6}$$

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[19] In (3) and (4) below, is contained the Painlevé transformation mentioned in the introduction.

[20] As also into $A_1, \cdots, A_{m-n}$ below.

We see now, because S is not divisible by A, that, for two distinct values t_1 and t_2 of t, with $t_2 > t_1$, (1) is the same except that the C_j for t_2 are those for t_1 multiplied by $S^{t_2-t_1}$.[16]

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6. Let the condition be satisfied. We shall prove that the general solution of A is an essential manifold.

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Consider, for $j = 2, \cdots, r$, the fractions, with positive denominators and positive values,

$$(2) \qquad \frac{\sum_{k=1}^{m-n} k i_{kj}}{p_j - p_1 + \sum_{k=1}^{m-n} i_{kj}}.$$

Let s be any number which exceeds each of these fractions.

We denote by c a positive real constant which will later be made small.[19] Let

$$(3) \qquad z = \frac{x - a}{c}.$$

For any $\tilde{y}$ approximating to $\bar{y}$ as in §6, A becomes a function of x, analytic at a and distinct from zero at a. Let

$$(4) \qquad w(z) = c^{-s} A$$

where $\tilde{y}$ is understood to be substituted into A[20] and where the x used in A is related to z as in (3). Then w is analytic at $z = 0$ and

$$w_1 = c^{-s+1} A_1, \cdots, w_{m-n} = c^{-s+m-n} A_{m-n}$$

where the subscripts of w indicate differentiation with respect to z. Let

$$D_j(z) = C_j$$

with $\tilde{y}$ substituted into C_j.

The j^{th} term in (1) goes over into

$$(5) \qquad c^{u_j} D_j(z)\, w^{p_j} w_1^{i_{1j}} \cdots w_{m-n}^{i_{m-n,j}}$$

where

$$u_j = s\left(p_j + \sum_{k=1}^{m-n} i_{kj}\right) - \sum_{k=1}^{m-n} k i_{kj}.$$

We have $u_1 = p_1 s$ and, referring to (2), we see that $u_j > p_1 s$ for $j > 1$.

As (1) vanishes for $y = \tilde{y}$, we can write

$$(6) \qquad D_1(z)\, w^{p_1} + \sum_{j=2}^{r} c^{u_j - p_1 s} D_j(z)\, w^{p_j} \cdots w_{m-n}^{i_{m-n,j}} = 0.$$

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[19] In (3) and (4) below, is contained the Painlevé transformation mentioned in the introduction.

[20] As also into $A_1, \cdots, A_{m-n}$ below.

Each term in $\sum$ of (6) contains a positive power of c.

Taking a definite $\tilde{y}$, which has been found for a definite μ and ϵ, we proceed to associate with it a definite value of c.

Let the Taylor expansion of $A(\tilde{y}, x)$[21] at a be

$$(7) \qquad b_0 + b_1(x - a) + \cdots + b_i(x - a)^i + \cdots .$$

Then $b_0 \neq 0$ but if, in §6, ϵ and $1/\mu$ are small, a large number of the b_i beginning with b_0 will be small. This is because $\tilde{y}$ approximates to $\bar{y}$ and $\bar{y}$ annuls A.

Let v be the greatest integer less than s. We choose c so as to make the greatest of the quantities

$$| b_i c^{-s+i} |, \qquad i = 0, 1, \cdots, v$$

equal to unity. This is possible because $b_0 \neq 0$.

Then c tends towards 0 with ϵ and $1/\mu$.

For $| z |$ small, we have by (3), (4), (7).

$$(8) \qquad w(z) = \sum_{i=0}^{\infty} b_i c^{-s+i} z^i .$$

When μ and $1/\epsilon$ increase, the coefficient of z^i in (8), for a fixed i exceeding v, will tend towards zero. For if $i > v$, then $i \geqq s$.

It follows that we can select a sequence of approximating solutions $\tilde{y}(x)$ for which $w(z)$ tends toward a non-vanishing polynomial of degree v at most, in the sense that each coefficient in (8) tends toward the corresponding coefficient in the polynomial.[22] Let $\gamma(z)$ be such a polynomial.

Let, for $| x - a |$ small, and for the particular solution $\bar{y}$,

$$C_j(\bar{y}, x) = \sum_{i=0}^{\infty} h_{ij}(x - a)^i, \qquad j = 1, \cdots, r.$$

Then

$$C_j(\tilde{y}, x) = \sum_{i=0}^{\infty} h'_{i\,j}(x - a)^i$$

where, for each i, $h'_{i\,j}$ approaches h_{ij} as ϵ and $1/\mu$ decrease. We have

$$D_j(z) = \sum_{i=0}^{\infty} h'_{i\,j} c^i z^i .$$

Turning now to (6), and remembering that all terms in $\sum$ of (6) contain positive powers of c, we recognize that

$$(9) \qquad h_{01}\, \gamma^{p_1} = 0.$$

In short, if the first member of (9) did not have a vanishing expansion in powers of z, the first member of (6) could not have a vanishing expansion when c is small and w approximates closely to γ.

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[21] The result of substituting $\tilde{y}$ into A.

[22] In this polynomial, the "coefficient of z^i" with $i > v$ is understood to be zero.

Because $C_1(\bar{y}, x)$ does not vanish at a, we have $h_{01} \neq 0$. Then, by (9), $\gamma = 0$. This contradicts what precedes, so that the sufficiency proof is completed.

REMARK. It might be proposed to treat the sufficiency question by making the substitution $A = \alpha^h$, with h a positive integer, in the relation $S^tF = 0$. For h sufficiently large the resulting relation could be divided through by $\alpha^{hp_1} = A^{p_1}$ and we would get a relation which could not be satisfied by a sequence of $\tilde{y}$ for which the Taylor expansions of α at a tend to vanish. Unfortunately there is no assurance that such a sequence of $\tilde{y}$ exists.

Digression on a Decomposition Problem

8. We know that F can be decomposed into irreducible systems by taking the system Φ composed of F and a sufficient number of its derivatives, and resolving Φ, considered as a set of simple forms, into indecomposable systems.[23] The method of §7 can be used to throw some light on the question as to how many derivatives of F are necessary.

Let the general solution of A be an essential manifold.

Let q be the greatest of the quantities (2). Let g be the smallest integer not less than p_1q where p_1 is as in §7.

Let $F, F_1, \cdots, F_g$, with F_j the j^{th} derivative of F, be considered as a system of simple forms. We resolve this system into essential indecomposable systems. There will be precisely one indecomposable system—call it Λ—which is not held by S_1, the separant of F. Let the forms in Λ be considered as differential polynomials, forming a system Σ. Then Σ holds the general solution of F.

We say that Σ *does not hold the general solution of* A.

Suppose that the contrary is true. Then if $\bar{y}$ is the solution of A used in §7,

$$\bar{y}, \bar{y}_1, \cdots, \bar{y}_{m+g} \tag{10}$$

is a solution of Λ. Because AS_1 does not hold Λ, there is an area in which we can approximate to (10) with arbitrary closeness by solutions of Λ for which AS_1 is distinct from zero throughout the area. Let a be any point of this area.

The solutions approximating to (10) furnish initial conditions for solutions $\tilde{y}$ in the general solution of F which do not annul A at a and for which

$$|\,\tilde{y}(a_j) - \bar{y}_j(a)\,|, \qquad j = 0, \cdots, m + g$$

are as small as one pleases.

For such solutions $\tilde{y}$, the first $g + m - n + 1$ coefficients in the expansion of A at a will be small. This is because the expansion of $\tilde{y}_i$, for $i = 0, \cdots, n$, approximates closely to that of $\bar{y}_i$ up to and including the term in $(x - a)^{g+m-n}$.

Let s of §7 exceed q so slightly that $v \leqq q$. In what follows, we keep μ of §7

[23] *A. D. E.*, §87.

fixed at the value $g + m$, while ϵ will be given a succession of values tending toward zero. We proceed as in §7. We note that

$$v \leqq q \leqq p_1 q \leqq g < g + m - n,$$

so that $b_0, \cdots, b_v$ are small with ϵ, and c approaches zero with ϵ.

We can thus select a sequence of w's for which the sum of the first $g + m - n + 1$ terms in the expansion (8) tends toward a non-vanishing polynomial γ of degree $d \leqq v$.

In the expansion of the first member of (6), the first $g + 1$ terms come from the first $g + m - n + 1$ terms of w and terms of the $D_j(z)$. Hence the sum of the first $g + 1$ terms in the expansion of the first member of (6) tends toward the corresponding sum for $h_{01}\gamma^{p_1}$. The latter sum is thus zero. Now

$$g + 1 \geqq p_1 q + 1 > p_1 v \geqq p_1 d,$$

so that $h_{01}\gamma^{p_1} = 0$. This contradiction proves our statement.

Example. Let $F = y_2^2 - y$. Then the only singular solution is $y = 0$. If we put $A = y$, (1) becomes $A_2^2 - A$, so that $y = 0$ is an essential manifold. Now $p_1 = 1$ and $q = 4$. Then $g = 4$. Thus, if we use A and its first four derivatives, we can, by an elimination, secure a set of equations which completely define the general solution of A.[24]

Necessity Proof

9. We assume now that, among the terms of lowest degree in (1), there is a term which involves one of the derivatives of A. We shall prove that the general solution of A is not an essential manifold.

Our procedure will be as follows. Let the essential irreducible manifolds in the manifold of F be the general solutions of forms $B_1, \cdots, B_d$. We shall produce a form E, of order at most n and not divisible by A, such that every solution in the general solution of A which does not annul E belongs to the general solution of some B_i of order exceeding n. It will then be easy to see that the general solution of A is not essential.

The first step in our work will be to present the form E. We shall then show that, given any $\bar{y}$ in the general solution of A which does not annul E, we can satisfy F *formally* by a series

$$y = \bar{y} + \varphi_1 c + \varphi_2 c^{\rho_2} + \cdots + \varphi_k c^{\rho_k} + \cdots \tag{11}$$

in which c is an arbitrary constant; the ρ_k rational numbers greater than unity, with a common denominator, which increase with k; and the φ_k functions of x, all analytic at some point a independent of k. (This does not mean that the φ_k have a common area of analyticity.) The series (11) will have only formal significance, and no investigation of its convergence properties or asymptotic properties will be made.[25] The derivatives of y in (11) are calculated formally.

[24] Actually only three derivatives are necessary. Cf. *A. D. E.*, §15.
[25] Cf. §90.

We shall see that (11) is a formal solution of one of the essential irreducible systems into which F decomposes. The coefficient φ_1 will have such generality as to require that the irreducible system just mentioned be the general solution of a form of order greater than n. Finally, it will be seen that $\bar{y}$ is a solution of this irreducible system.

10. We introduce a new unknown u_0 and denote the j^{th} derivative of u_0 by u_{0j}.

We are going to replace y in (1) by $y + u_0$ and to examine the resulting form in y and u_0. Such a replacement made in any form K in y, of order s, will convert K into

$$K + K_0 u_{00} + \cdots + K_s u_{0s} + \text{terms of higher degree in the } u_{0i}, \tag{12}$$

where K_i is the partial derivative of K with respect to y_i.

For $K = A$, (12) will contain the term Su_{0n} and for $K = A_i$ (12) will contain $Su_{0,n+i}$.

We see from (12) that if y is specialized as a solution $\bar{y}$ of A, the j^{th} term of (1) becomes a form in u_0 whose terms of lowest degree in $u_{00}, u_{01}, \cdots, u_{0m}$ are of degree at least

$$p_j + i_{1j} + \cdots + i_{m-n,j}.$$

Considering (1) as a polynomial in A and the A_i, we take its terms of *lowest* degree and select from them those terms which are of a *highest* degree (possibly 0) in A_{m-n}. From the terms just taken, we select those for which the degree in A_{m-n-1} is highest. We continue in this manner, through A_1. Our process isolates a single term

$$C_j A^{p_j} A_1^{i_{1j}} \cdots A_{m-n}^{i_{m-n,j}} \tag{13}$$

of (1).

Suppose now that $\bar{y}$ is in the general solution of A but does not annul C_jS. Then (13) becomes a form in u_0 whose terms of lowest degree include the term

$$C_j(\bar{y})S^g(\bar{y})\, u_{0n}^{p_j} u_{0,n+1}^{i_{1j}} \cdots u_{0m}^{i_{m-n,j}} \tag{14}$$

with $g = p_j + i_{1j} + \cdots + i_{m-n,j}$. The term (14) cannot be cancelled by the terms in u_0 coming from any term other than (13) in (1).

C_jS will serve as the form E mentioned in §9. For the substitution $y = \bar{y} + u_0$, (1) goes over into a form in u_0 which we shall denote by H. The coefficients in H need not belong to the field underlying our work. H will contain no term free of u_0 and its derivatives,[26] and (14) will be one of the terms of lowest degree in H. We observe that (14) is of order higher than n in u.

11. We are going to determine, for H, a formal solution of the type

$$u_0 = \varphi_1 c + \varphi_2 c^{\rho_2} + \cdots + \varphi_k c^{\rho_k} + \cdots. \tag{15}$$

[26] Because $\bar{y}$ annuls F.

with c an arbitrary constant; the ρ_k rational numbers greater than unity, with a common denominator, which increase with k; and the φ_k functions of x, all analytic at some point a.[27]

It should be emphasized that we find only *certain* solutions (15), rather than all such solutions.

Let $L(u_0)$[28] be the sum of the terms of lowest degree in H. If (15) satisfies $H = 0$ formally, we must have

$$L(\varphi_1) = 0. \tag{16}$$

We note that L involves derivatives of u_0, in fact derivatives of order higher than n, so that (16) has solutions distinct from zero. Let φ_1 be any such solution.

It may be that $\varphi_1 c$ causes H to vanish identically in x and c. If so, $\varphi_1 c$ is a series (15) such as we are seeking. In what follows, we assume that $\varphi_1 c$ is not a solution.

Then $H(\varphi_1 c)$ is a polynomial in c in which the lowest exponent of c exceeds the degree of L in $u_{00}, \cdots, u_{0m}$.

We put $u_0 = \varphi_1 c + u_1$ in H.[29] Then H goes over into an expression H' in x, c, u_1. We arrange H' as a polynomial in $u_{10}, \cdots, u_{1m}$. Then H' can be written

$$H' = a'(c) + \sum b_i'(c)\, u_{10}^{\alpha_{0i}}\, u_{11}^{\alpha_{1i}} \cdots u_{1m}^{\alpha_{mi}}. \tag{17}$$

Here $a'(c)$ and the $b'(c)$ are polynomials in c with coefficients which are analytic functions of x. The terms in $\sum$ are those which are not free of $u_{10}, \cdots, u_{1m}$ and we understand that no b_i' is zero. We know that $a'(c) = H(\varphi_1 c)$ is not zero and that its least exponent of c exceeds the degree of L. As to i, it ranges from unity to some positive integer.

Let σ' be the least exponent of c in a' and σ_i' the least exponent of c in b_i'.[30] Let

$$\rho_2 = \operatorname{Max} \frac{\sigma' - \sigma_i'}{\alpha_{0i} + \cdots + \alpha_{mi}} \tag{18}$$

where i has the range $1, 2, \cdots$, which it has in $\sum$.

We shall prove that $\rho_2 > 1$.

Every power product in $u_{00}, \cdots, u_{0m}$ in H of degree at least unity contributes a like power product to $\sum$ in (17) and the corresponding b_i' will have a σ_i' equal

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[27] In what follows, we parallel the Newton polygon method for algebraic functions as presented in Hensel and Landsberg, *Algebraische Funktionen*, Chapters IV, V, to the degree which analogies permit.

[28] In representing forms, we shall sometimes display only the unknown, and at other times derivatives as well. In each case our meaning will be clear.

[29] Differentiation of unknowns u, will be indicated by a second subscript.

[30] Of course, we consider only exponents effectively present.

to zero. Thus, in choosing the maximum in (18), we have to consider quantities

$$\frac{\sigma' - \sigma_i'}{\alpha_{0i} + \cdots + \alpha_{mi}} \tag{19}$$

for which σ_i' is zero and the denominator is the degree of L. As σ' exceeds the degree of L, we have $\rho_2 > 1$.

12. Let g' represent the coefficient of $c^{\sigma'}$ in a'. Let h_i' denote the coefficient of $c^{\sigma_i'}$ in b_i', or denote 0, according as (19) equals ρ_2 or is less than ρ_2. Let[31]

$$L'(u_1) = g' + \sum h_i' u_{10}^{\alpha_{0i}} \cdots u_{1m}^{\alpha_{mi}}. \tag{20}$$

Then L' is not free of $u_{10}, \cdots, u_{1m}$. We determine an analytic φ_2 as any solution of

$$L'(\varphi_2) = 0, \tag{21}$$

subject to the following stipulation.

Let L' be of effective degree $f \geqq 1$ in $u_{10}, \cdots, u_{1m}$. Then certain partial derivatives of L' with respect to $u_{10}, \cdots, u_{1m}$, of orders f, will be non-vanishing functions of x. Let then $f_1 \leqq f$ be the smallest positive integer such that we can find a solution of (21) which does not annul some partial derivative of L' of order f_1. We understand in what follows that φ_2 fails to annul some partial derivative of order f_1.[32]

13. It may be that $\varphi_2 c^{\rho_2}$ causes H' to vanish in c and x. Then

$$u_0 = \varphi_1 c + \varphi_2 c^{\rho_2}$$

is a solution of H of the type (15). Let us suppose that the vanishing does not occur.

We make, in H', the substitution

$$u_1 = \varphi_2 c^{\rho_2} + u_2. \tag{22}$$

Then H' goes over into an expression H'' in x, c, u_2 which can be written

$$H'' = a''(c) + \sum b_i''(c) u_{20}^{\alpha_{0i}} \cdots u_{2m}^{\alpha_{mi}}. \tag{23}$$

Here a'' and the b'' are sums in which each term is the product of a rational power of c and a function of x. We know that $a'' \neq 0$ and we understand that no b'' vanishes. The sums $\sum$ in (17) and (23) do not necessarily involve the same power products.

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[31] Cf. (26) below for motivation.

[32] Note that no corresponding condition was put on φ_1.

Let σ'' be the least exponent of c in a'' and σ_i'' the least exponent in b_i''. Let

$$(24) \qquad \rho_3 = \text{Max}\, \frac{\sigma'' - \sigma_i''}{\alpha_{0i} + \cdots + \alpha_{mi}}.$$

We are going to prove that $\rho_3 > \rho_2$.

Let ψ be an indeterminate which admits of differentiation with respect to x. We shall replace u_1 in H' by $c^{\rho_2}\psi$. The i^{th} term in $\sum$ of (17) will produce a set of terms, each of the type

$$\beta c^q \psi^{\alpha_{0i}} \cdots \psi_m^{\alpha_{mi}}$$

with β a function of x and with

$$(25) \qquad q \geqq \sigma_i' + \rho_2(\alpha_{0i} + \cdots + \alpha_{mi}).$$

By (18), $q \geqq \sigma'$. We will have $q = \sigma'$ only if β is an h_i' in (20). On this basis, we may write

$$(26) \qquad H'(c^{\rho_2}\psi) = c^{\sigma'}L'[\psi, \cdots, \psi_m] + c^{\tau'}M'[c, \psi, \cdots, \psi_m],$$

where, relative to L', M', τ', the following statements apply.

L' is as in (20), with u_1 replaced by ψ. M' is a polynomial in $\psi, \cdots, \psi_m$ whose coefficients are sums in which each term is the product of a non-negative rational power of c by a function of x. As to τ', which we understand to be taken as large as possible, it is a rational number greater than σ'.

We now put $\psi = \varphi_2 + c^{-\rho_2}u_2$. Then (26) gives, by (22),[33]

$$(27) \qquad \begin{aligned} H''(u_2) = {} & c^{\sigma'}L'[\varphi_{20} + c^{-\rho_2}u_{20}, \cdots, \varphi_{2m} + c^{-\rho_2}u_{2m}] \\ & + c^{\tau'}M'[c, \varphi_{20} + c^{-\rho_2}u_{20}, \cdots, \varphi_{2m} + c^{-\rho_2}u_{2m}]. \end{aligned}$$

Suppose that, for some set of non-negative integers $l_0, \cdots, l_m$,

$$(28) \qquad \frac{\partial^{l_0 + \cdots + l_m} L'(u_1)}{\partial^{l_0} u_{10} \cdots \partial^{l_m} u_{1m}}$$

does not vanish for $u_1 = \varphi_2$.[34] This implies that at least one l is not zero. We shall see that $u_{20}^{l_0} \cdots u_{2m}^{l_m}$ is present in (23) and we shall determine the σ_i'' associated with that power product.

The coefficient of $u_{20}^{l_0} \cdots u_{2m}^{l_m}$, for any $l_0, \cdots, l_m$, in the second member of (27) is the quotient by $l_0! \cdots l_m!$ of

$$(29) \qquad \begin{aligned} & c^{\sigma' - \rho_2(l_0 + \cdots + l_m)} L'_{l_0 \cdots l_m}(\varphi_{20}, \cdots, \varphi_{2m}) \\ & + c^{\tau' - \rho_2(l_0 + \cdots + l_m)} M'_{l_0 \cdots l_m}(c, \varphi_{20}, \cdots, \varphi_{2m}), \end{aligned}$$

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[33] Differentiation of φ_2 is indicated by a second subscript.

[34] $L'(u_1)$ is as in (20).

where $L'_{l_0 \cdots l_m}$ is (28) with u_1 replaced by φ_2 and where

$$M'_{l_0 \cdots l_m} = \frac{\partial^{l_0 + \cdots + l_m}}{\partial \psi^{l_0} \cdots \partial \psi^{l_m}} M(c, \psi, \cdots, \psi_m)$$

with ψ replaced by φ_2.

Our hypothesis[35] as to φ_2 implies thus that

$$\sigma''_i = \sigma' - \rho_2(l_0 + \cdots + l_m). \tag{30}$$

On the other hand, if φ_2 annuls (28), then, if $u_{20}^{l_0} \cdots u_{2m}^{l_m}$ is present in $\sum$ in (23), we shall have, for the associated σ''_i,

$$\sigma''_i > \sigma' - \rho_2(l_0 + \cdots + l_m). \tag{31}$$

We can now study ρ_3 in (24). We have, for any i, representing $\alpha_{0i} + \cdots + \alpha_{mi}$ by β_i,

$$\frac{\sigma'' - \sigma''_i}{\beta_i} = \frac{\sigma'' - \sigma'}{\beta_i} + \frac{\sigma' - \sigma''_i}{\beta_i}. \tag{32}$$

By (30) and (31) we have

$$\frac{\sigma' - \sigma''_i}{\beta_i} = \rho_2 \tag{33}$$

or

$$\frac{\sigma' - \sigma''_i}{\beta_i} < \rho_2 \tag{34}$$

according as (28) with $l_j = \alpha_{ji}$ does not vanish or does vanish. From (33) and (34) we see that $(\sigma' - \sigma''_i)/\beta_i$ is a maximum, namely ρ_2, for those i for which (28) does not vanish. Such i exist, as was seen in connection with the stipulation made above in regard to φ_2.

From (29) with every l zero, we see that $\sigma'' > \sigma'$. Turning now to (32), we see that there are i for which the first member of (32) exceeds ρ_2.

This proves that $\rho_3 > \rho_2$.

14. We now form, for H'', an equation

$$L''(\varphi_3) = 0$$

analogous to (21) and take φ_3 with a stipulation similar to that made for φ_2.

We continue this procedure. It may be that at some stage we reach an $H^{(k-1)}$ which is annulled by $\varphi_k c^{\rho_k}$. In that case

$$u = \varphi_1 c + \cdots + \varphi_k c^{\rho_k}$$

[35] The hypothesis that (28) does not vanish.

is a solution of H of the type (15). We suppose, in what follows that our process does not terminate in a finite number of steps, so that we are led to form an infinite series of the type (15).

We shall prove that the ρ_k have a common denominator. This will mean that the ρ_k become infinite with k. We shall prove also that (15) annuls H. One sees easily that the φ_k can be taken so as to have a common point of analyticity.

15. We begin by proving that the degrees of the $L^{(k)}$ as polynomials in the u_{ki} do not increase with k. Let us compare the degree of L' with that of L''. Let f and $f_1 \leqq f$ be as in the stipulation relative to φ_2, with f the degree of L'.

In (32), $(\sigma'' - \sigma')/\beta_i$ is less for $\beta_i > f_1$ than for $\beta_i \leqq f_1$. On the other hand, $(\sigma' - \sigma_i'')/\beta_i$ attains its maximum value ρ_2 for some β_i equal to f_1. This shows that the first member of (32) cannot be as great as ρ_3 for $\beta_i > f_1$. We see now, referring to the description of the coefficients in (20), which description is similar to that of the coefficients in L'', that the degree of L'' does not exceed f_1.

Thus there is an integer e such that, for $k \geqq e$, the $L^{(k)}$ are all of the same degree, say s. Consider any $k \geqq e$. Given any solution φ of $L^{(k)}$, every partial derivative of $L^{(k)}$ of order less than s must vanish for $u_k = \varphi$.

Certain partial derivatives of $L^{(k)}$ of order $s - 1$ are effectively of the first degree. Let Q be one of them. Then Q holds $L^{(k)}$. It follows that if $L^{(k)}$ is resolved into irreducible factors in the domain of rationality of its coefficients, every irreducible factor whose order in u_k is not less than that of Q is the product of Q by a function of x.[36] Let then

$$L^{(k)} = Q^j R$$

with R of lower order than Q. Suppose that $j < s$. Let p be the order of Q. Let φ annul Q but not R. We have

$$\frac{\partial^j L^{(k)}}{\partial u_{kp}^j} = \mu R \tag{35}$$

with μ a function of x. Thus the first member of (35) is not annulled by φ. We infer that

$$L^{(k)} = \lambda Q^s \tag{36}$$

with λ a function of x. In the expression of $L^{(k)}$ analogous to (20), there is a term like g' in (20), free of $u_{k0}, \cdots, u_{km}$. This means that Q has such a term, so that, by (36), $L^{(k)}$ has terms of the first degree. Thus, in the relation for ρ_{k+1} analogous to (18), there will be, among those i which give a maximum, certain i for which

$$\alpha_{0i} + \cdots + \alpha_{mi} = 1.$$

[36] Cf. *A. D. E.* §21.

In other words, the denominator of the rational number ρ_{k+1} can be taken as the common denominator of $\sigma^{(k)}$ and the $\sigma_i^{(k)}$. For that common denominator, we can use the common denominator of $\rho_2, \cdots, \rho_k$.

This shows that the ρ_k have a common denominator, so that they approach ∞ with k.

16. It remains to show that (15) annuls H. Because $a^{(k)}$, for any k,[37] is the result of substituting the first k term of (15) into H, it suffices to prove that $\sigma^{(k)}$ approaches ∞ with k. That is so because the $\sigma^{(k)}$ increase with k and have a common denominator.

17. We have thus established the existence of the solutions (11) of F. In forming such solutions, we can take φ_1 as any solution of the differential equation (16), which equation is of order greater than n. Let F be decomposed into essential irreducible systems $\Sigma_1, \cdots, \Sigma_d$. Given a solution (11), it must be a solution of some Σ_i. Otherwise, let R_i, $i = 1, \cdots, d$, be a form in Σ_i which does not vanish for the solution. Now $T = R_1 \cdots R_d$ holds F. Thus some power of T is a linear combination of F and its derivatives with forms for coefficients. We obtain the contradiction that T vanishes for (11).

Let the manifold of Σ_i, $i = 1, \cdots, d$, be the general solution of a form B_i. We say that there is a solution (11) which is a solution of some Σ_i whose B_i is of order higher than n. Suppose that this is not so. Let $\Sigma_1, \cdots, \Sigma_p$ be those Σ_i whose B_i have orders not exceeding n. Then every solution (11) annuls $T = B_1 \cdots B_p$. Under the substitution $y = \bar{y} + u_0$, T goes over into a form $V(u_0)$ which is annulled by every (15). Then φ_1 must annul the sum of the terms of lowest degree in V. As those terms are of order at most n and as φ_1 is any solution[38] of an equation of order higher than n, we have a contradiction.

Thus there is a Σ_j with B_j of order higher than n which admits some solution (11). Let R be any form in Σ_j. If R goes over into $D(u_0)$ for $y = \bar{y} + u_0$, then $D(u_0) = 0$ for $u_0 = 0$. Otherwise (15) could not annul D. This means that $R(\bar{y}) = 0$, so that $\bar{y}$ is a solution of Σ_j.

We shall now complete the proof that the general solution of A is not essential. Let $\Sigma_1, \cdots, \Sigma_p$ be those Σ_i among $\Sigma_1, \cdots, \Sigma_d$ whose B_i are of order higher than n. We say that the general solution of A is part of the manifold of some Σ_i with $i \leqq p$. Let this be false, and let R_i in Σ_i, $i = 1, \cdots, p$, not hold the general solution of A. Then $T = ER_1 \cdots R_p$, with E as in §10, does not hold the general solution of A. But we saw above that if $E(\bar{y}) \neq 0$, some R_i is annulled by $\bar{y}$. This completes the proof of the sufficiency of the condition stated in §6.

A Special Case

18. On the basis of what precedes, we can record the following special result.

Let F be algebraically irreducible.[39] *Let $y = 0$ be a solution of F. Let the terms*

[37] Cf. (23).

[38] $\varphi_1 = 0$ gives the solution $y = \bar{y}$ of F, of type (11).

[39] This condition is not essential and can easily be removed.

is a solution of H of the type (15). We suppose, in what follows that our process does not terminate in a finite number of steps, so that we are led to form an infinite series of the type (15).

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of lowest degree in F involve a y_p with $p \geqq 1$. Then $y = 0$ is not an essential manifold in the manifold of F and $y = 0$ is contained in an essential irreducible manifold which is the general solution of a form of order at least p.

PART II. EQUATIONS OF SECOND ORDER

Generalities

19. We deal with a form F in y which is of the second order, and algebraically irreducible. Our problem is to study the distribution of the singular solutions of F among the irreducible manifolds in the manifold of F.

Considering the system

$$F, S, \tag{37}$$

where S is the separant of F, our first step is to decompose (37) into irreducible systems by the method of *A. D. E.*, Chapter V. We obtain algebraically irreducible forms, none divisible by any other,

$$B_1, \cdots, B_p; \qquad C_1, \cdots, C_q, \tag{38}$$

where the B are of the first order in y and the C of order zero, such that the manifold of (37) is made up of the general solutions of the B and of the manifolds of the C.

The general solutions of the B will necessarily be essential manifolds for (37). The first question which arises is whether the C are all essential. This amounts to asking, for every i and j, whether the manifold of C_i is contained in the general solution of B_j. One would determine first whether B_j holds C_i. If B_j holds C_i, the theorem of §5 decides whether the manifold of C_i is contained in the general solution of B_j. It may thus be assumed, after certain C are suppressed, that the manifolds of the C are essential. Then no C will have a solution in common with the general solution of any B.[40]

The theorem of §5 enables us also to determine the singular solutions of any B_i which belong to the general solution of that B_i.

We have thus an accurate separation of the manifold of (37), that is, of the singular solutions of F, into irreducible manifolds. One might wish in addition to know whether, for some i and j with $i \neq j$, the general solutions of B_i and B_j have solutions in common. The manifold of the system B_i, B_j consists of the manifolds of certain irreducible forms of order zero. These irreducible forms may be tested, as above, to see whether their manifolds are in the general solutions of B_i and B_j.

The above simple classification of the singular solutions of F leaves out of consideration the relation of the singular solutions to the general solution of F. There arises the problem of determining those singular solutions which belong to

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[40] If C_i has a solution in common with a form D, the remainder of D with respect to C_i, which is of lower rank than C_i, will admit the solution and will therefore vanish identically. Then D holds C_i.

the general solution of F. The theorem of §5 permits the determination of those C in (38) whose manifolds are essential in the manifold of F and of those B whose general solutions are essential. The remaining C have manifolds which belong to the general solution of F[41] and the remaining B have general solutions which belong to the general solution of F. This settles the question as to which irreducible manifolds in the decomposition of (37) belong to the general solution of F. Suppressing those forms in (38) whose general solutions[42] are not essential for F, we have the manifold of F decomposed into the general solutions (all essential) of forms[43]

$$F;\ B_1, \cdots, B_r;\ C_1, \cdots, C_s. \tag{39}$$

20. We come now to the raison d'être for our investigation of equations of order 2. The question arises as to whether the general solution of B_i in (39), for any particular i, has any solutions in common with the general solution of F. A form D_i of order zero can be determined which is necessarily annulled by such common solutions.[44] The problem then amounts to the following: *Given an irreducible form of order zero,*[45] *is the manifold of the form contained in the general solution of F?*

The form of order zero may, without loss of generality, be taken simply as y. The reduction of the general case to this special one is made in §88. Our problem may thus be formulated: *Let F vanish for $y = 0$. It is required to determine whether $y = 0$ is a solution in the general solution of F.*

The important case, of course, is that in which $y = 0$ is not an essential manifold and in which $y = 0$ belongs to the general solution of one or more B in (39).[46]

Series Solutions

21. Let us suppose that $y = 0$ is in the general solution of B_1. Then, according to §5, if B_1 is considered as a polynomial in y and y_1, y_1 must be present in one or more of the terms of lowest degree.

Let the degree of B_1 in y_1 be n. If we consider the relation $B_1 = 0$ as an algebraic relation between y_1 and y, $B_1 = 0$ admits n solutions y_1, each of the form

$$y_1 = \varphi_1(x)y^{\rho_1} + \cdots + \varphi_k(x)y^{\rho_k} + \cdots \tag{40}$$

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[41] No C has its manifold in the general solution of any B.

[42] The general solution of a C_i is the manifold of C_i.

[43] $r \leqq p$, $s \leqq q$.

[44] Putting $A = B_i$, we consider (1). As the manifold of A is essential, a solution common to the general solutions of A and F must annul C_1 of §7. The resultant of C_1 and A with respect to y_1 can be taken as D_i.

[45] Evidently D_i can be replaced by a set of irreducible forms each of which has its manifold in the general solution of B_i above. It is somewhat more convenient at this point to leave the B out of the discussion.

[46] If $y = 0$ is not an essential manifold and is not in the general solution of any B, $y = 0$ belongs to the general solution of F.

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the general solution of F. The theorem of §5 permits the determination of those C in (38) whose manifolds are essential in the manifold of F and of those B whose general solutions are essential. The remaining C have manifolds which belong to the general solution of F[41] and the remaining B have general solutions which belong to the general solution of F. This settles the question as to which irreducible manifolds in the decomposition of (37) belong to the general solution of F. Suppressing those forms in (38) whose general solutions[42] are not essential for F, we have the manifold of F decomposed into the general solutions (all essential) of forms[43]

$$F;\, B_1, \cdots, B_r;\, C_1, \cdots, C_s. \tag{39}$$

20. We come now to the raison d'être for our investigation of equations of order 2. The question arises as to whether the general solution of B_i in (39), for any particular i, has any solutions in common with the general solution of F. A form D_i of order zero can be determined which is necessarily annulled by such common solutions.[44] The problem then amounts to the following: *Given an irreducible form of order zero,*[45] *is the manifold of the form contained in the general solution of F?*

The form of order zero may, without loss of generality, be taken simply as y. The reduction of the general case to this special one is made in §88. Our problem may thus be formulated: *Let F vanish for $y = 0$. It is required to determine whether $y = 0$ is a solution in the general solution of F.*

The important case, of course, is that in which $y = 0$ is not an essential manifold and in which $y = 0$ belongs to the general solution of one or more B in (39).[46]

Series Solutions

21. Let us suppose that $y = 0$ is in the general solution of B_1. Then, according to §5, if B_1 is considered as a polynomial in y and y_1, y_1 must be present in one or more of the terms of lowest degree.

Let the degree of B_1 in y_1 be n. If we consider the relation $B_1 = 0$ as an algebraic relation between y_1 and y, $B_1 = 0$ admits n solutions y_1, each of the form

$$y_1 = \varphi_1(x)y^{\rho_1} + \cdots + \varphi_k(x)y^{\rho_k} + \cdots \tag{40}$$

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[41] No C has its manifold in the general solution of any B.

[42] The general solution of a C_i is the manifold of C_i.

[43] $r \leqq p$, $s \leqq q$.

[44] Putting $A = B_i$, we consider (1). As the manifold of A is essential, a solution common to the general solutions of A and F must annul C_1 of §7. The resultant of C_1 and A with respect to y_1 can be taken as D_i.

[45] Evidently D_i can be replaced by a set of irreducible forms each of which has its manifold in the general solution of B_i above. It is somewhat more convenient at this point to leave the B out of the discussion.

[46] If $y = 0$ is not an essential manifold and is not in the general solution of any B, $y = 0$ belongs to the general solution of F.

where the ρ_k are rational numbers, with a common denominator, which increase with k, and where the φ_k are functions of x all analytic in some area. These power series solutions can be determined by the Newton polygon method.

We wish to show that either there is a solution (40) of the type $y_1 = 0$ or else that, for some solution (40), $\rho_1 \geqq 1$.[47] If B_1 is the product of y_1 by a function of x, the only solution (40) is $y_1 = 0$. Otherwise, let B_1 be considered as a polynomial in y_1. If y_1^i is actually present in B_1, let σ_i denote the least exponent of y in the coefficient of y_1^i. Because B_1 is algebraically irreducible, there is a σ_0. According to the Newton polygon method, the greatest possible ρ_1 in (40) is the greatest value of

$$\frac{\sigma_0 - \sigma_i}{i} \tag{41}$$

where i ranges over all of its positive values. Because y_1 occurs among the terms of lowest degree in B_1 considered as a polynomial in y, y_1, there will be at least one $i > 0$ for which $\sigma_0 \geqq \sigma_i + i$. For such an i, (41) is not less than unity.

We examine a solution (40) with $\rho_1 \geqq 1$. The second member of (40) is analytic in x and y for y small and distinct from zero, x remaining in a suitable area. If for a suitable x, we attribute to y a small value distinct from zero, then (40), considered as a differential equation, will define a function $y(x)$ belonging to the general solution of B_1, hence to the manifold of F. The successive derivatives of such a function $y(x)$ can be obtained from (40) by termwise differentiation. In particular, we find for $y_2(x)$,

$$y_2 = \varphi_{11}y^{\rho_1} + \cdots + \varphi_{k1}y^{\rho_k} + \cdots + y_1(\rho_1\varphi_1 y^{\rho_1 - 1} + \cdots)$$

and, using (40), we have

$$y_2 = \psi_1 y^{\tau_1} + \cdots + \psi_k y^{\tau_k} + \cdots \tag{42}$$

where the τ_k have the same denominator as the ρ_k and increase with k, and where $\tau_1 \geqq 1$. We secure similar series for the higher derivatives of $y(x)$, each series, if it does not vanish, beginning with a term in y of exponent at least unity.

Because of the arbitrariness of the initial value of $y(x)$, it must be that the second members of (40) and (42), when substituted into F for y_1 and y_2, annul F identically in x and y.

22. Let F now be any algebraically irreducible form in y, of the second order, which vanishes for $y = 0$. We make no assumptions as to the types of irreducible manifolds in the manifold of F or as to the irreducible manifolds to which $y = 0$ belongs.

We shall consider formal differential equations of the type

$$y_1 = \varphi_1 y^{\rho_1} + \cdots + \varphi_k y^{\rho_k} + \cdots \tag{43}$$

[47] If the second member of (40) is not zero, we understand that $\varphi_1 \neq 0$.

where the ρ_k are as above, with $\rho_1 \geqq 1$, and where the φ_k are functions of x all analytic at some point a. We do not assume that the φ_k have a common *area* of analyticity and, what is then natural, we make no assumption as to the convergence of the series in (43). We differentiate (43) formally, securing a series for y_2 as in (42). If the series for y_1 and y_2 annul F,[48] we shall call the series in (43) a *y-solution* of F.[49]

We have seen that y-solutions exist in the case in which there are B to whose general solutions $y = 0$ belongs.

We are going to investigate a large class of the y-solutions of F. It will turn out that $y = 0$ belongs to the general solution of F when and only when F has a y-solution which does not annul any B which may be present in (39).[50] The investigation of the y-solutions will furnish a method for determining, by a finite number of operations, whether or not $y = 0$ is in the general solution of F.

A Partial Differential Equation

23. Let F be as in §22 and let it admit the series in (43) as a y-solution. Let u represent the series in (43). If y_2 is the series in (42), we have

$$y_2 = uu_y + u_x, \tag{44}$$

the partial derivatives being taken formally. Thus the series u formally satisfies the partial differential equation

$$H(x, u, u_x, u_y) = 0 \tag{45}$$

obtained by replacing y_1 and y_2 in $F = 0$ by u and $uu_y + u_x$ respectively.

On this basis, we call u a y-solution of H. As F and H have the same y-solutions, it will serve our purposes to investigate the y-solutions of H.

Multiplicities

24. A y-solution of H will be said to be of multiplicity p, where p is a positive integer if

$$\frac{\partial H}{\partial u}, \frac{\partial^2 H}{\partial u^2}, \cdots, \frac{\partial^{p-1} H}{\partial u^{p-1}}$$

vanish for the y-solution, but $\partial^p H/\partial u^p$ does not.[51]

A y-solution of H of multiplicity p will, considered as a y-solution of F, also be said to have multiplicity p.

It must not be thought that the above definition provides a multiplicity for

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[48] Naturally, it is assumed that a lies in the area in which the coefficients in F are meromorphic.

[49] In §90, we give an example of a y-solution which diverges for every $y \neq 0$.

[50] When there are no B the existence of a y-solution of F is necessary and sufficient for $y = 0$ to belong to the general solution of F. Of course, when there are no B, the theorem of §5 tells whether $y = 0$ is in the general solution.

[51] For $p = 1$, this means simply that $\partial H/\partial u$ does not vanish.

every y-solution. Thus, if $F = y_2$, then $H = uu_y + u_x$ and every $\partial^k H/\partial u^k$ vanishes for $u = 0$.

Sets of Solutions

25. Let F be as in §22. We represent by $\mathfrak{A}$ the area in which the coefficients in F are meromorphic.

Let n denote a positive integer. We shall call n the *y-solution number* of F if there exists a set of points $\mathfrak{S}$,[52] contained in $\mathfrak{A}$ and having no limit point in the interior of $\mathfrak{A}$, such that, given any simply connected area $\mathfrak{A}_1$ in $\mathfrak{A}$ which contains no point of $\mathfrak{S}$, F has a finite set of distinct[53] y-solutions which satisfy the following conditions:

(a) *The coefficients in the y-solutions are analytic throughout $\mathfrak{A}_1$.*

(b) *Each y-solution has a multiplicity, and the sum of the multiplicities, for the y-solutions of the set, is n.*

(c) *Every y-solution of F with coefficients analytic at a point in $\mathfrak{A}_1$ coincides with some y-solution of the set.*

It is evident that if n exists as above, n is unique.

26. If F has y-solutions and if no n exists as above, we shall call ∞ the y-solution number of F. The following situation will be shown then to exist.

There exists a rational number $\rho \geqq 1$ such that, given any area $\mathfrak{A}_1$ in $\mathfrak{A}$, there exists an area $\mathfrak{A}_2$, contained in $\mathfrak{A}_1$, which has the following properties. F has a certain infinite set of y-solutions, each of which has all its coefficients analytic at some point in $\mathfrak{A}_2$.[54] *For $\rho_i \leqq \rho$, the coefficient of a term in y^{ρ_i} in any of these y-solutions is analytic throughout $\mathfrak{A}_2$. Each of these y-solutions has a term in y^ρ and any two of them coincide in their terms of degree less than ρ in y, but differ in their terms of degree ρ.*

It is easy to see that the situation just described cannot be realized when n exists as in §25.

If F has no y-solutions, we shall define the y-solution number of F as 0.

The formal counterpart of what precedes is as follows. When the y-solution number is infinite, we shall, in a certain process, meet a differential equation. In the finite case we encounter only algebraic equations.

Theorems on Solution Numbers

27. We formulate some theorems whose proofs will occupy many sections. We take F as any algebraically irreducible form, of the second order, which vanishes for $y = 0$.

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[52] $\mathfrak{S}$ may be vacuous.

[53] Wherever we have occasion to refer to y-solutions as distinct or coincident, our meaning will be that which is natural under the circumstances and will be clear without special comment.

[54] The point may be different for different y-solutions.

Let F be considered as a polynomial in y, y_1, y_2.

THEOREM I: *If one or more of the terms of lowest degree in F involve y_2, the y-solution number of F is ∞.*

THEOREM II: *Let no term of lowest degree in F involve y_2, while one or more such terms involve y_1. Let n be the greatest exponent of y_1 in the terms of lowest degree. Then the y-solution number of F is n or ∞.*

THEOREM III: *If the terms of lowest degree in F involve neither y_2 nor y_1, the y-solution number of F is 0.*

Theorems I and III are not important as far as applications go,[55] but they form a natural complement to Theorem II.

Polygons

28. We shall have occasion to consider various expressions in x, y and two other letters v_1, v_2 of the type

$$\sum_{i=1}^{r} a_i y^{\alpha_i} v_1^{\beta_i} v_2^{\gamma_i} \tag{46}$$

where the α_i are non-negative rational numbers and the β_i, γ_i non-negative integers. The a_i are functions of x distinct from zero.

Let G be an expression (46). We are going to form a polygon for G, of the Newton type.

For each i, let $\lambda_i = \beta_i + \gamma_i$. In a plane referred to rectangular axes, we plot the points (λ_i, α_i), $i = 1, \cdots, r$. We secure thus r or fewer points, each point being associated with one or more terms of G.

We consider those of the plotted points which have a least abscissa—say the abscissa ζ_1—and choose from them that point which has a least ordinate—say the ordinate σ_1. For all points (λ_i, α_i) with $\lambda_i > \zeta_1$, if such exist, we form the ratio

$$\frac{\sigma_1 - \alpha_i}{\zeta_1 - \lambda_i} \tag{47}$$

which is the slope of the straight segment joining (ζ_1, σ_1) to (λ_i, α_i). Let us suppose that there are segments whose slopes (47) do not exceed -1. Taking those segments whose slope is a minimum, we choose the longest of them. Let its right extremity be denoted by (ζ_2, σ_2).

Arithmetically, what we have done amounts to taking the least ratio (47), assumed not to exceed -1, and picking the largest $\lambda_i (= \zeta_2)$ which gives a least ratio.

It may be that there are points with $\lambda_i > \zeta_2$ for which

$$\frac{\sigma_2 - \alpha_i}{\zeta_2 - \lambda_i} \leqq -1. \tag{48}$$

[55] The hypothesis of Theorem I implies directly that $y = 0$ is in the general solution of F. That of Theorem III implies that $y = 0$ is an essential manifold.

If so, we take those points which minimize the first member of (48) and choose from them that point (ζ_3, σ_3) whose abscissa ζ_3 is the greatest. When (ζ_2, σ_2) is joined to (ζ_3, σ_3) we secure a segment which, seen from (ζ_2, σ_2), slants downward and has a greatest possible angle of depression, the angle being at least $\frac{1}{4}\pi$.

We continue this construction as long as it is possible to secure such angles of depression. The polygon formed by the segments obtained will be called the *polygon of G*.

On a geometric basis, it is obvious that each segment after the first has a smaller angle of depression, that is, a greater slope, than its predecessor.

If there are no points with $\lambda_i > \zeta_1$ or if there are no such points for which (47) does not exceed -1, the polygon of G is defined as the point (ζ_1, σ_1).

When we speak of the points (λ_i, α_i) lying on a side of a polygon, the extremities of the side will be included.

By the *polygon of F* (F as in §27), we mean the polygon of the expression obtained from F putting $y_1 = v_1$, $y_2 = v_2$.

A point (λ_i, α_i), plotted for G and lying on the polygon of G, will be called an *a-point* or a *b-point* according as there are not or are terms involving v_2 associated with the point

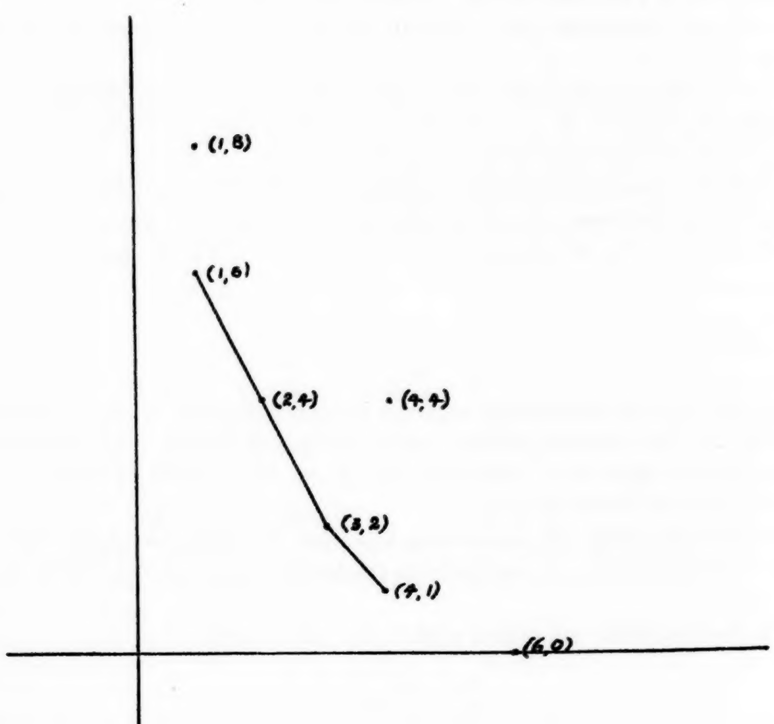

FIG. 1

Example 1: Let

$$G = y^6v_1 + y^8v_2 + y^4v_1v_2 + y^2v_1^3 + y^2v_1^2v_2 + yv_1^4 + y^4v_2^4 + v_1^6.$$

The (λ_i, α_i) and the polygon are shown in Fig. 1. The point $(3,2)$ is associated with two terms, namely $y^2v_1^3$ and $y^2v_1^2v_2$. The points $(1,6)$ and $(4,1)$ are a-points, while $(2,4)$ and $(3,2)$ are b-points.

Example 2. Let

$$G = y^2v_1 + y^2v_2 + v_1^5.$$

The polygon consists of the point $(1,2)$, which is a b-point.

Polygon of H.

29. Let H of §§23, 24 be written

$$H = \sum_{i=1}^{r} a_i y^{\mu_i} u^{\beta_i} u_x^{\gamma_i} u_y^{\delta_i} \tag{49}$$

where the a_i are functions of x distinct from zero. Let

$$\alpha_i = \mu_i - \delta_i, \qquad \lambda_i = \beta_i + \gamma_i + \delta_i.$$

We plot all points (λ_i, α_i) and form a polygon, which may consist of a single point, exactly as in §28. This polygon will be called the *polygon of H*.

A point (λ_i, α_i), plotted for one or more terms of H and lying on the polygon of H, will be called an *a-point*, or a *b-point*, according as there are not, or are, terms involving at least one of u_x, u_y associated with the point.

30. Let $\mathcal{P}$ be the polygon of F and $\mathcal{Q}$ that of H. We are going to compare $\mathcal{P}$ and $\mathcal{Q}$. It will turn out that $\mathcal{P}$ and $\mathcal{Q}$ are identical,[56] except in special cases. Our principal object is to compare the b-points of $\mathcal{Q}$ with those of $\mathcal{P}$.

We consider first the case in which $\mathcal{P}$ has no b-points.

Let us suppose first that $\mathcal{P}$ consists of a single point, (ζ_1, σ_1). As (ζ_1, σ_1) is an a-point, it comes from a single term in F, a term in $y^{\sigma_1} y_1^{\zeta_1}$. The corresponding term in $y^{\sigma_1} u^{\zeta_1}$ in H[57] will contribute the point (ζ_1, σ_1) to the set of points which we have to plot for H.

Consider any other term of F, in $y^{\alpha}y_1^{\beta}y_2^{\gamma}$. This produces a point (λ, α), where $\lambda = \beta + \gamma$, which either lies above (ζ_1, σ_1), or else, when joined to that point, gives a segment of slope exceeding -1. The corresponding terms in H come from

$$y^{\alpha}u^{\beta}(uu_y + u_x)^{\gamma}. \tag{50}$$

[56] This adjective will be made precise.

[57] It is easy to see that this term is not cancelled.

They will yield the points

$$(51) \qquad (\lambda + p, \alpha - p), \qquad p = 0, 1, \cdots, \gamma.$$

All of these points lie on a line sloping downward from (λ, α) at an angle of $\frac{1}{4}\pi$. One of these points, namely (λ, α), may possibly lie vertically above (ζ_1, σ_1). Any of the points which is not so situated will yield, when joined to (ζ_1, σ_1), a segment of slope exceeding -1. Thus $\mathcal{Q}$ is the single point (ζ_1, σ_1).

Suppose now that $\mathcal{P}$ consists of a single side. What precedes shows that $\mathcal{P}$ and $\mathcal{Q}$ have the same first point (ζ_1, σ_1). Consider a point (λ, α), plotted for F, which is not on $\mathcal{P}$. It will have related with it, among the points for H, points of the type (51). None of these points, joined to (ζ_1, σ_1), will yield a slope as small as that of the side of $\mathcal{P}$. On the other hand, every a-point of $\mathcal{P}$[58] is a point plotted for H. This shows that $\mathcal{P}$ and $\mathcal{Q}$ are identical, carrying the same plotted points, which are a-points for both of them.

Continuing in this way,[59] we see that when $\mathcal{P}$ has only a-points, $\mathcal{Q}$ is identical with $\mathcal{P}$.

The whole situation can be grasped intuitively as follows. Let us visualize $\mathcal{P}$, together with all points plotted for F not on $\mathcal{P}$. The points plotted for H are obtained by adding, to those for F, points on 45° lines sloping downward from points which, for every side of $\mathcal{P}$, lie in the upper half-plane determined by that side. Such new points cannot influence the formation of $\mathcal{Q}$.

We now examine the case in which $\mathcal{P}$ has b-points. First let (ζ_1, σ_1), the leftmost point of $\mathcal{P}$, be a b-point, and let a term in $y^\alpha y_1^\beta y_2^\gamma$, $\gamma > 0$, be associated with (ζ_1, σ_1). Then H will have a term in $y^\alpha u^\beta u_x^\gamma$, which will give (ζ_1, σ_1) as the leftmost point, indeed a b-point, of $\mathcal{Q}$. Suppose now that (ζ, σ), with $\zeta > \zeta_1$, associated with $y^\alpha y_1^\beta y_2^\gamma$, $\gamma > 0$, is the leftmost b-point on $\mathcal{P}$. Among the vertices (extremities of sides) of $\mathcal{P}$ which lie to the left of (ζ, σ), let (ζ_i, σ_i) be the rightmost. Then, from (ζ_1, σ_1) through (ζ_i, σ_i), $\mathcal{P}$ and $\mathcal{Q}$ are identical. The term in $y^\alpha u^\beta u_x^\gamma$ in H gives the point (ζ, σ). If a point plotted for F yields points (51) of abscissa exceeding ζ_i, such points (51), joined to (ζ_i, σ_i) give slopes no less than that of the segment from (ζ_i, σ_i) to (ζ, σ). Thus (ζ, σ) is on $\mathcal{Q}$ and is a b-point for $\mathcal{Q}$. We observe that no term involving u_y is associated with (ζ, σ).[60]

A closer examination[61] would show that, except for two cases, $\mathcal{P}$ and $\mathcal{Q}$ are identical (that is, have the same a-points and the same b-points), and that the b-points of $\mathcal{Q}$ are associated with terms involving u_x but not u_y. The two exceptions are:

[58] In fact, every point plotted for F.

[59] For the case of two sides, the first side of $\mathcal{P}$ is seen as above to be a first side of $\mathcal{Q}$. We have then to consider points (51) whose abscissas exceed ζ_2 of §28. Such points, joined to (ζ_2, σ_2), will give slopes greater than that of the second side of $\mathcal{P}$.

[60] No point plotted for F of abscissa less than σ can yield a point (51) coinciding with (ζ, σ).

[61] Not important for what follows.

(a) If $\mathcal{P}$ terminates in a side of slope -1 which has b-points, $\mathcal{P}$ and $\mathfrak{Q}$ will coincide in all their sides, except perhaps their last sides. The final side of $\mathfrak{Q}$, also of slope -1, may have on it b-points which are either a-points for $\mathcal{P}$ or do not occur on $\mathcal{P}$.

(b) If the final side of $\mathcal{P}$ is of slope less than -1, and if its right extremity is a b-point, $\mathfrak{Q}$ will consist of $\mathcal{P}$ and of a final side, of slope -1, adjoined to $\mathcal{P}$.

A Polygon Process

31. We are going to show that *if $\mathfrak{Q}$ has b-points the y-solution number of F is ∞.*

Let us show that this will imply the truth of Theorem I. Lef F have a term of lowest degree involving $y^\alpha y_1^\beta y_2^\gamma$ with $\gamma > 0$. Let $\lambda = \beta + \gamma$. Suppose that (λ, α) is not on $\mathcal{P}$. Let (ζ, σ) be the rightmost point of $\mathcal{P}$. Then $\zeta + \sigma$ is the degree of a term of F, so that

$$\zeta + \sigma \geqq \lambda + \alpha. \tag{52}$$

Then $\lambda \neq \zeta$, for if λ equaled ζ, the point (λ, α), since it is not on $\mathcal{P}$, would have to lie above (ζ, σ) and we would have $\alpha > \sigma$, a contradiction of (52). Suppose that $\lambda < \zeta$. Then (λ, α) lies above $\mathcal{P}$, that is, a half-line directed downward from (λ, α) intersects $\mathcal{P}$. Then the slope from (λ, α) towards (ζ, σ) is less than -1. This contradicts (52). Finally, let $\lambda > \zeta$. Then the slope from (ζ, σ) towards (λ, α) exceeds -1, a contradiction of (52). We conclude that (λ, α) is a b-point on $\mathcal{P}$. Then $\mathfrak{Q}$ must have a b-point (§30). This proves our statement relative to Theorem I.

32. If $\mathfrak{Q}$ has a b-point, $\mathfrak{Q}$ cannot consist of a single point.[62] We therefore consider the *side* l of $\mathfrak{Q}$, carrying the *leftmost* b-point of $\mathfrak{Q}$. This leftmost b-point is not associated with terms involving u_y (§30).

We write

$$H = L + M \tag{53}$$

where L consists of those terms of H which are associated with points on l.

Let $-\rho_1$ be the slope of l, where $\rho_1 \geqq 1$. Suppose that, in H, we put

$$u = w_0 y^{\rho_1}, \qquad u_x = w_1 y^{\rho_1}, \qquad u_y = w_2 y^{\rho_1 - 1} \tag{54}$$

where w_0, w_1, w_2 are indeterminates. Then a term

$$a y^\mu u^\beta u_x^\gamma u_y^\delta \tag{55}$$

in H goes over into

$$a w_0^\beta w_1^\gamma w_2^\delta y^{(\mu-\delta)+\rho_1(\beta+\gamma+\delta)}. \tag{56}$$

[62] Let (ζ, σ), a b-point, be the only point on $\mathfrak{Q}$. Then (ζ, σ) is a b-point of $\mathcal{P}$, so that F has a term in $y^\sigma y_1^\alpha y_2^\beta$ with $\alpha + \beta = \zeta$, $\beta > 0$. This term produces, for H, points on a line of slope -1, sloping downward from (ζ, σ). We secure the contradiction that $\mathfrak{Q}$ has more than one point.

The exponent of y in (56) is the intercept, on the axis of ordinates, of a line of slope $-\rho_1$ passing through the point associated with (55). This intercept will have a common value, say σ, for all terms in L in (53) and will exceed σ for all terms in M.[63] For the substitution (54), L goes over into

$$y^\sigma R(x, w_0, w_1, w_2)$$

with R a polynomial in the w_i whose coefficients are functions of x. Also M goes over into

$$y^\tau S(x, y, w_0, w_1, w_2)$$

where S is a polynomial in the w_i whose coefficients are sums of terms ay^μ with a a function of x and μ rational and non-negative; and where τ, which we take as large as possible, is a rational number greater than σ. Then

$$H(x, y, w_0 y^{\rho_1}, w_1 y^{\rho_1}, w_2 y^{\rho_1 - 1}) = y^\sigma R(x, w_0, w_1, w_2) + y^\tau S(x, y, w_0, w_1, w_2). \tag{57}$$

33. Consider the differential equation for an unknown function φ_1 of x,

$$R(x, \varphi_1, \varphi_{11}, \rho_1 \varphi_1) = 0 \tag{58}$$

where R is as in (57) and where φ_{11} is the derivative of φ_1. We wish to show that φ_{11} figures in (58). The leftmost b-point on l yields, when (54) is used, a set of distinct terms of the type $ay^\sigma w^p w_1^q$, with $q > 0$ in some terms and with $p + q$ the abscissa of the b-point. These produce, for (58), terms $a\varphi_1^p \varphi_{11}^q$ which we shall show are not cancelled. Any other point on l produces, for (58), terms $a\varphi_1^p \varphi_{11}^q$ with $p + q$ the abscissa of the point. This shows that the cancellation is impossible.

We take φ_1 as any solution of (58). Given any area $\mathfrak{A}_1$ contained in $\mathfrak{A}$ of §25, there is an $\mathfrak{A}_2$ in $\mathfrak{A}_1$ in which an infinite number of such solutions are analytic.

34. If $\varphi_1 y^{\rho_1}$, with φ_1 the special function just determined, annuls H, then $\varphi_1 y^{\rho_1}$ is a y-solution of H. In what follows, we assume that H is not so annulled. We put, in H,

$$u = \varphi_1 y^{\rho_1} + u_1. \tag{59}$$

Then H goes over into an expression H' in u_1, u_{1x}, u_{1y} which is not annulled by $u_1 = 0$. We write

$$H' = a' + \sum b_i' u_1^{\beta_i} u_{1x}^{\gamma_i} u_{1y}^{\delta_i}. \tag{60}$$

Here a' and the b' are sums (non-vanishing) of terms ay^μ with a a function of x and μ rational and non-negative. The least exponent of y in such a sum will be called the *order* of the sum. The range of i is from unity to some positive integer.

[63] Points plotted for H which are not on l lie in the upper half-plane determined by l produced.

We represent by σ' the order of a' and, by σ_i', the result of subtracting δ_i from the order of b_i'.

Let ρ_2 be the greatest of the quantities

$$\frac{\sigma' - \sigma_i'}{\beta_i + \gamma_i + \delta_i}. \tag{61}$$

One of the objects of what follows is to prove that $\rho_2 > \rho_1$.

35. Let ψ be an indeterminate which can be differentiated partially with respect to x and y. We put, in H,

$$u = \psi y^{\rho_1}, \qquad u_x = \psi_x y^{\rho_1}, \qquad u_y = y^{\rho_1 - 1}(\rho_1 \psi + y\psi_y). \tag{62}$$

Then, by (57),

$$H(\psi y^{\rho_1}) = y^\sigma R(x, \psi, \psi_x, \rho_1\psi + y\psi_y) + y^\tau S(x, y, \psi, \psi_x, \rho_1\psi + y\psi_y). \tag{63}$$

We put

$$\psi = \varphi_1 + y^{-\rho_1} u_1. \tag{64}$$

Then (63) gives

$$H'(u_1) = y^\sigma R(x, \varphi_1 + y^{-\rho_1}u_1, \varphi_{11} + y^{-\rho_1}u_{1x}, \rho_1\varphi_1 + y^{-\rho_1+1}u_{1y}) + y^\tau S(x, y, \cdots). \tag{65}$$

Let us suppose that for certain non-negative integers l_0, l_1, l_2,

$$\frac{\partial^{l_0+l_1+l_2} R(x, w_0, w_1, w_2)}{\partial w_0^{l_0} \partial w_1^{l_1} \partial w_2^{l_2}} \tag{66}$$

does not vanish for

$$w_0 = \varphi_1, \qquad w_1 = \varphi_{11}, \qquad w_2 = \rho_1\varphi_1. \tag{67}$$

This implies that l_0, l_1, l_2 are not all zero. We shall show that $u_1^{l_0} u_{1x}^{l_1} u_{1y}^{l_2}$ is present in $\sum$ in (60) and we shall calculate the corresponding σ_i'.

For any l_0, l_1, l_2, whether (66) is zero or not, the coefficient of $u_1^{l_0} u_{1x}^{l_1} u_{1y}^{l_2}$ in the second member of (65) is the quotient by $l_0!\, l_1!\, l_2!$ of

$$y^{\sigma + l_2 - \rho_1(l_0+l_1+l_2)} R_{l_0 l_1 l_2}(x, \varphi_1, \varphi_{11}, \rho_1\varphi_1) + y^{\tau + l_2 - \rho_1(l_0+l_1+l_2)} S_{l_0 l_1 l_2}(x, y, \varphi_1, \varphi_{11}, \rho_1\varphi_1). \tag{68}$$

The R term in (68) is (66) with the substitution (67) performed. The S term has a similar meaning.

Our hypothesis as to the non-vanishing of (66) implies that[64]

$$\sigma_i' = \sigma - \rho_1(l_0 + l_1 + l_2). \tag{69}$$

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[64] One should recall the meaning of σ_i'.

On the other hand, if (66) vanishes, then, if $u_1^{l_0} u_{1x}^{l_1} u_{1y}^{l_2}$ is present in $\sum$, we will have, for the associated σ_i',

$$\sigma_i' > \sigma - \rho_1(l_0 + l_1 + l_2). \tag{70}$$

We can now study ρ_2. We have, for any i, letting $\lambda_i = \beta_i + \gamma_i + \delta_i$,

$$\frac{\sigma' - \sigma_i'}{\lambda_i} = \frac{\sigma' - \sigma}{\lambda_i} + \frac{\sigma - \sigma_i'}{\lambda_i}. \tag{71}$$

By (69) and (70) we have

$$\frac{\sigma - \sigma_i'}{\lambda_i} = \rho_1 \tag{72}$$

or

$$\frac{\sigma - \sigma_i'}{\lambda_i} < \rho_1 \tag{73}$$

according as (66), with $l_0 = \beta_i$, $l_1 = \gamma_i$, $l_2 = \delta_i$ does not vanish or does vanish when (67) holds. The last term in (71) thus has ρ_1 for its maximum value.[65]

By (68) with $l_0 = l_1 = l_2 = 0$, we see that $\sigma' > \sigma$. It follows from (71) that there are i for which the first member of (71) exceeds ρ_1. This proves that $\rho_2 > \rho_1$.

36. The expression (60) of H' can be written in the form of the expression of H in (49), with the difference that we may have to use fractional μ_i. We construct a polygon $\mathfrak{Q}'$ for H', exactly as in §29, and define *a-point* and *b-point* as in §29.

From §34 we see that the leftmost point on $\mathfrak{Q}'$ is $(0, \sigma')$. The quantities (61) are negatives of slopes of segments joining $(0, \sigma')$ to points plotted for H'. The least such slope is $-\rho_2$, which is less than -1. Thus $\mathfrak{Q}'$ has at least one side, and its first side has $-\rho_2$ for slope.

37. We wish to see under what circumstances $\mathfrak{Q}'$ can have b-points. Towards this end, we examine the relation between F and H'.

Let $\xi = \varphi_1 y^{\rho_1}$. Then H' is obtained from H by the substitution $u = \xi + u_1$. Thus H' comes from F by the substitution

$$y_1 = \xi + u_1, \qquad y_2 = \xi\xi_y + \xi_x + (u_{1x} + \xi_y u_1 + \xi u_{1y} + u_1 u_{1y}). \tag{74}$$

Let

$$v_1 = u_1, \qquad v_2 = u_{1x} + \xi_y u_1 + \xi u_{1y} + u_1 u_{1y}. \tag{75}$$

Let

$$G = F(x, y, \xi + v_1, \xi\xi_y + \xi_x + v_2).$$

Then H' is found from G by means of (75).

[65] It is easy to see that there are l_i for which (66) does not vanish.

Let a polygon $\mathcal{P}'$ be constructed for G as in §28. Consider a term of G in $y^\alpha v_1^\beta v_2^\gamma$. The corresponding point plotted for G is (λ, α) where $\lambda = \beta + \gamma$. Consider, for H', the terms coming from

$$y^\alpha u_1^\beta (u_{1x} + \xi_y u_1 + \xi u_{1y} + u_1 u_{1y})^\gamma. \tag{76}$$

A typical term obtained from (76), written without its numerical coefficient is

$$y^\alpha u_1^\beta u_{1x}^{p_1} (\xi_y u_1)^{p_2} (\xi u_{1y})^{p_3} (u_1 u_{1y})^{p_4} \tag{77}$$

where $p_1 + p_2 + p_3 + p_4 = \gamma$. We write (77)

$$y^\alpha \xi_y^{p_2} \xi^{p_3} u_1^{\beta+p_2+p_4} u_{1x}^{p_1} u_{1y}^{p_3+p_4}. \tag{78}$$

The exponents of y in ξ_y and ξ are $\rho_1 - 1$ and ρ_1 respectively. Then the point plotted for H' for (78) is

$$(\lambda + p_4, \alpha - p_4 + (p_2 + p_3)(\rho_1 - 1)). \tag{79}$$

For $p_2 = p_3 = p_4 = 0$, (79) is (λ, α). Also if $\rho_1 = 1$ and $p_4 = 0$, (79) is (λ, α). Any point (79) distinct from (λ, α) either is vertically above (λ, α) or else, when joined to that point, produces a slope not less than -1.

One sees now, as in §30, that $\mathfrak{Q}'$ has b-points only when $\mathcal{P}'$ does. If $\mathcal{P}'$ has b-points, the leftmost such point—call it p—will be a b-point (and the leftmost one) for $\mathfrak{Q}'$. There is one term of H' associated with p to which we wish to call particular attention. Among the terms of G associated with p, let $ay^\alpha v_1^\beta v_2^\gamma$ have a maximum degree in v_2. Then (76) will yield for H' a term

$$ay^\alpha u_1^\beta u_{1x}^\gamma.$$

Any other term of H' associated with p has a degree in u_{1x} less than γ. A term associated with a point whose abscissa exceeds that of p has a degree in u_1, u_{1x}, u_{1y} which exceeds $\beta + \gamma$. These facts will be important in §38.

38. We return to the formation of a y-solution of H. Using the first side of $\mathfrak{Q}'$ as the side l of $\mathfrak{Q}$ was used in §32, we put, in H',

$$u_1 = w_0 y^{\rho_2}, \qquad u_{1x} = w_1 y^{\rho_2}, \qquad u_{1y} = w_2 y^{\rho_2 - 1}. \tag{80}$$

The intercept on the axis of ordinates of the first side of $\mathfrak{Q}'$ is σ'. We have thus a relation, analogous to (57),

$$H'(x, y, w_0 y^{\rho_2}, \cdots) = y^{\sigma'} R'(x, w_0, w_1, w_2) + y^{\tau'} S' \tag{81}$$

with $\tau' > \sigma'$.

We determine a function $\varphi_2(x)$ as a solution of

$$R'(x, \varphi_2, \varphi_{21}, \rho_2 \varphi_2) = 0 \tag{82}$$

which has an area of analyticity in common with φ_1, the solution being subject to a stipulation which will be made in a moment.

It is important to know that (82) is not free of φ_2 and φ_{21}. This is certainly

true if w_2 does not appear in R' in (81). If w_2 appears, the final remarks of §37 show that there is a term $aw_0^\beta w_1^\gamma$ which will yield effectively, for (82), a term $a\varphi_2^\beta\varphi_{21}^\gamma$.

Now for the stipulation. Let f be the degree in w_0, w_1, w_2 of

$$R'(x, w_0, w_1, w_2). \tag{83}$$

Then certain partial derivatives of (83) with respect to w_0, w_1, w_2, of order f, do not vanish when

$$w_0 = \varphi_2, \qquad w_1 = \varphi_{21}, \qquad w_2 = \rho_2\varphi_2, \tag{84}$$

for any solution of (82). Let f_1 be the smallest integer for which a solution φ_2 of (82) exists such that some derivative of (83) of order f_1 does not vanish for (84). We understand φ_2 to be so taken that some derivative of order f_1 does not vanish.

39. Now let g be the degree in w_0, w_1, w_2 of R in (57). We shall show that the degree of R' in (81) does not exceed g. We inspect (71). For certain $\lambda_i \leqq g$, $(\sigma - \sigma_i')/\lambda_i$ attains its maximum value ρ_1. Now $(\sigma' - \sigma)/\lambda_i$ is less for $\lambda_i > g$ than for $\lambda_i \leqq g$. This shows that (61) cannot be a maximum for

$$\beta_i + \gamma_i + \delta_i > g.$$

Now every term in R' comes from a term in H' which is associated with a point on the first side of $\mathfrak{Q}'$. To a term in H' of this type, in

$$y^\mu u_1^\beta u_{1x}^\gamma u_{1y}^\delta,$$

with $\beta + \gamma + \delta > 0$, there corresponds a maximum quantity (61) with $\sigma_i' = \mu - \delta, \beta_i = \beta, \gamma_i = \gamma, \delta_i = \delta$. This proves our statement.

40. If $\varphi_2 y^{\rho_2}$ annuls $H'(u_1)$ then

$$u = \varphi_1 y^{\rho_1} + \varphi_2 y^{\rho_2}$$

is a y-solution of H. Otherwise, we put

$$u_1 = \varphi_2 y^{\rho_2} + u_2$$

and $H'(u_1)$ goes over into an expression $H''(u_2)$. We give H'' the treatment accorded to H' in §§34–39. Considerations perfectly parallel to those of §§34, 35 furnish a $\rho_3 > \rho_2$. To (63), there corresponds a relation

$$H'(\psi y^{\rho_2}) = y^{\sigma'} R'(x, \psi, \psi_x, \rho_2\psi + y\psi_y) + y^{\tau'} S'. \tag{85}$$

We discuss the b-points on the polygon of H'' as in §37, with the difference that now

$$\xi = \varphi_1 y^{\rho_1} + \varphi_2 y^{\rho_2}$$

and that, when p_2 and p_3 are not both zero in (78), we get, for (78), in addition to the point (79), one or more points lying vertically above that point. Using a relation analogous to (81), we secure an equation similar to (82)

$$R''(x, \varphi_3, \varphi_{31}, \rho_3\varphi_3) = 0, \tag{86}$$

and take φ_3 with a stipulation similar to that in §38.

In the continuation of this process, it may be that we reach an $H^{(k-1)}$ which is annulled by $\varphi_k y^{\rho_k}$. If so,

$$u = \varphi_1 y^{\rho_1} + \cdots + \varphi_k y^{\rho_k}$$

is a y-solution of H.

We assume, in what follows, that we are led to an infinite series

$$\varphi_1 y^{\rho_1} + \cdots\cdot + \varphi_k y^{\rho_k} + \cdots. \tag{87}$$

The φ_k are taken so as to have a common point of analyticity.

41. We shall prove that the ρ_k in (87) have a common denominator. This will show that the ρ_k increase toward ∞.

There is an integer e such that, for $k \geqq e$, the expressions

$$R^{(k)}(x, w_0, w_1, w_2) \tag{88}$$

analogous to (83) have a common degree, say s, in w_0, w_1, w_2.[66] We choose a $k \geqq e$. Then every partial derivative of $R^{(k)}$ of order less than s vanishes for[67]

$$w_0 = \varphi, \qquad w_1 = \varphi', \qquad w_2 = \rho_{k+1}\varphi \tag{89}$$

if φ satisfies

$$R^{(k)}(x, \varphi, \varphi', \rho_{k+1}\varphi) = 0. \tag{90}$$

If $R^{(k)}$ in (88) involves w_2, it involves w_1. (§38.) Let $R^{(k)}$ involve w_1. Among the terms involving w_1, let $w_0^{l_0} w_1^{l_1} w_2^{l_2}$ be one of highest total degree. Then

$$\frac{\partial^{l_0+l_1+l_2-1} R^{(k)}}{\partial w_0^{l_0}\, \partial w_1^{l_1-1}\, \partial w_2^{l_2}} \tag{91}$$

contains a single term involving w_1. This term is of the first degree in w_1 and is free of w_0, w_2. If we make the substitution (89) in the expression (91) and equate the result to zero, we obtain a differential equation

$$T(x, \varphi, \varphi') = 0$$

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[66] The degree of R'' does not exceed f_1 in §38. Cf. §15.

[67] φ' is the derivative of φ.

satisfied by every solution of (90). In T, φ' appears in a single term, a term which is free of φ. Every irreducible factor of the first member of (90) which involves φ' is T multiplied by a function of x.[68] Let, identically in x, φ, φ',

$$R^{(k)}(x, \varphi, \varphi', \rho_{k+1}\varphi) = UT^j \tag{92}$$

where U is free of φ'. We are interested in the result of making the substitution (89) in

$$\frac{\partial^j R^{(k)}(x, w_0, w_1, w_2)}{\partial w_1^j}. \tag{93}$$

The same result is obtained on putting $w_1 = \varphi'$ in

$$\frac{\partial^j}{\partial w_1^j} U(T(x, \varphi, w_1))^j. \tag{94}$$

Now (94) equals

$$\mu U$$

where μ is a function of x. Suppose that $j < s$. If we take φ so as to annul $T(x, \varphi, \varphi')$ but not U, we shall have a solution of (90) for which (93), with $j < s$, does not vanish. Thus $j = s$. Then U is free of φ, and we have an identity

$$R^{(k)}(x, \varphi, \varphi', \rho_{k+1}\varphi) = \nu T^s \tag{95}$$

with ν a function of x. $R^{(k)}$ in (95) is not annulled by $\varphi = 0$. This means that T contains a term in x alone, so that T^s contains a term which is of the first degree in φ' and free of φ. Then (88) must have a term which is of the first degree in w_1 and free of w_0, w_2.

It follows that for some maximum quantity

$$\frac{\sigma^{(k)} - \sigma_i^{(k)}}{\beta_i + \gamma_i + \delta_i}$$

we have $\gamma_i = 1, \beta_i = 0, \delta_i = 0$. Then the denominator of ρ_{k+1} can be taken as the common denominator of $\sigma^{(k)}$ and the $\sigma_i^{(k)}$. For that common denominator, we can use the common denominator of $\rho_1, \cdots, \rho_k$.

Suppose now that $R^{(k)}$ in (88) is free of w_1, w_2. It follows easily that

$$R^{(k)}(x, \varphi, \varphi', \rho_{k+1}\varphi) = \mu(\varphi - \nu)^s$$

with μ and ν functions of x and $\nu \neq 0$. This shows that $R^{(k)}$ in (88) has a term of the first degree in w_0 and we see as above that the denominator of ρ_{k+1} can be taken as the common denominator of $\rho_1, \cdots, \rho_k$.

42. As $\sigma^{(k)}$ increases with k and as the $\sigma^{(k)}$ have a common denominator, $\sigma^{(k)}$ must become infinite with k. Since $a^{(k)}$ (analogous to a' in (60)) is the result of substituting into H the sum of the first k terms in (87), (87) is a y-solution of H.

[68] Cf. §15.

This completes the proof of the result stated at the head of §31.

43. It must not be imagined that, for every y-solution of H, ρ_1 is the negative of the slope of some side of $\mathfrak{Q}$. For instance, let $F = y_2$. For every positive integer p,

(96) $$y = c^p + c^{p+1}x$$

with c constant is a solution of F. For (96) we have

$$y_1 = c^{p+1}.$$

Also, (96) gives

$$c = y^{1/p} + \mu y^{2/p} + \cdots$$

where $\mu, \cdots$, are functions of x. Then

(97) $$y_1 = y^{(p+1)/p} + \varphi_2 y^{(p+2)/p} + \cdots$$

and (97) is a y-solution of F. Thus there are an infinite number of possibilities for ρ_1. This cannot happen when $\mathfrak{Q}$ has no vertex which is a b-point.

Proofs of Theorems II and III

44. We investigate the case in which $\mathfrak{Q}$ has no b-points.

According to §31, y_2 does not occur among the terms of lowest degree in F.

Let the vertices[69] of $\mathfrak{Q}$, arranged according to increasing abscissas, be

(98) $$(\zeta_1, \sigma_1), \cdots, (\zeta_r, \sigma_r).$$

Let n be the greatest exponent of y_1 in the terms of lowest degree in F considered as a polynomial in y, y_1, y_2.[70] We shall prove that $\zeta_r = n$.

As $\mathfrak{Q}$ has no b-points, $\mathfrak{P}$ and $\mathfrak{Q}$ are identical. Let F have a term of lowest degree involving $y^p y_1^n$. Suppose that $n < \zeta_r$. Then (n, p) is either on $\mathfrak{P}$ or above $\mathfrak{P}$, so that

$$\frac{p - \sigma_r}{n - \zeta_r} \leqq -1.$$

Then $p + n \geqq \sigma_r + \zeta_r$. As F has a term in $y^{\sigma_r} y_1^{\zeta_r}$, $p + n = \sigma_r + \zeta_r$. Then n is not the highest exponent of y_1 in the terms of lowest degree. If $n > \zeta_r$, we see easily that $\mathfrak{P}$ does not terminate at (ζ_r, σ_r). Thus $n = \zeta_r$.

Because y_2 does not occur in the terms of lowest degree in F, u_x and u_y do not occur in the terms of lowest degree in H considered as a polynomial in y, u, u_x, u_y. The highest exponent of u in the terms of lowest degree in H is $n = \zeta_r$.

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[69] If $\mathfrak{Q}$ consists of a single point, that point will be called a vertex.

[70] n may be zero.

45. Suppose that, in (98), $\zeta_1 \geqq 1$. Then $u = 0$ is a y-solution of H. We shall prove that the multiplicity of this y-solution is ζ_1. Consider a term in H in

$$y^{\mu} u^{\beta} u_x^{\gamma} u_y^{\delta}. \tag{99}$$

If $\gamma + \delta > 0$, the partial derivatives of (99) with respect to u, of all orders, vanish for $u = 0$. If $\beta < \zeta_1$, we have $\gamma + \delta > 0$.

For any term in H involving u to a power greater than ζ_1, the first ζ_1 derivatives with respect to u vanish for $u = 0$. Finally, H contains terms in y and u alone, of degree ζ_1 in u, of the sum of which the first $\zeta_1 - 1$ derivatives vanish when $u = 0$, but not the derivative of order ζ_1.

46. Let $r = 1$ in (98). We are going to show that H has no y-solution except, possibly, $u = 0$. Let us assume the existence of a y-solution $\bar{u} \neq 0$ of H and let us substitute $\bar{u}$ into a term of H in

$$y^{\mu} u^{\beta} u_x^{\gamma} u_y^{\delta}. \tag{100}$$

There will result a series in y which will vanish only if $\bar{u}_x = 0$ and $\gamma > 0$. Otherwise, the lowest power of y obtained will equal

$$\mu - \delta + \rho_1(\beta + \gamma + \delta) \tag{101}$$

if φ_1 is not a constant and will exceed (101) if $\gamma \neq 0$ and φ_1 is a constant. Now (101) is the intercept on the axis of ordinates of a line of slope $-\rho_1$ passing through the point associated with (100). Because $\rho_1 \geqq 1$, this intercept is less for $y^{\zeta_1} u^{\sigma_1}$ than for any other expression (100). This proves our statement.

If $r = 1$ and $\zeta_1 = 0$, $u = 0$ does not satisfy H. This proves Theorem III.

If $r = 1$ and $\zeta_1 \geqq 1$, $u = 0$ is the only y-solution of H, and its multiplicity is $\zeta_1 = n$.

47. Suppose now that, in (98), $r > 1$. We are going to determine the possibilities for the first term $\varphi_1 y^{\rho_1}$ in a y-solution of H distinct from zero.

We prove first that ρ_1 must be the negative of the slope of a side of $\mathfrak{Q}$. Suppose that this is not so. If there are sides of $\mathfrak{Q}$ of slopes greater than $-\rho_1$, let (ζ_j, σ_j) be the first point from the left in (98) which is the extremity of such a side. Otherwise, let $j = r$. We consider a line through (ζ_j, σ_j) of slope $-\rho_1$. Then all points plotted for H other than (ζ_j, σ_j) lie above this line. It follows, as in §46, that the y-solution of H whose existence was assumed does not annul H.

48. Now let ρ_1 be a number such that $-\rho_1$ is the slope of some side l of $\mathfrak{Q}$. We form the relation (57) of §32. R will be free of w_1, w_2. We write it $R(x, w_0)$. It is easy to see[71] now that φ_1 must be a solution, distinct from zero, of

$$R(x, \varphi_1) = 0. \tag{102}$$

[71] Note that $y^{\sigma} R(x, \varphi_1)$ is the sum of the lowest powers of y obtained from the terms of H putting $u = \varphi_1 y^{\rho_1} + \cdots$.

The degree of (102) in φ_1 is the abscissa of the right extremity of l. The number of solutions of (102) which are distinct from zero equals the length of the horizontal projection of l.

49. We have thus formed two conditions which the first term in a y-solution must satisfy. We now undertake to prove that these conditions are sufficient and to determine the possibilities for the terms after the first.

In working, below, with a solution φ_1 of (102), we stay in some definite area in which φ_1 is analytic. Later we shall use the obvious fact that φ_1 is analytic in $\mathfrak{A}$ of §25, except for poles and isolated branch points of finite order.

For any $\varphi_1 y^{\rho_1}$ with φ_1 and ρ_1 as in §48, we form $H'(u_1)$ as in §34. Now, however, H' may vanish for $u_1 = 0$. We examine $\mathfrak{Q}'$, particularly those sides, if any, whose slopes are less than $-\rho_1$.

Let φ_1 be a solution of (102) of multiplicity p. Then (69), which holds even if H' vanishes for $u_1 = 0$,[72] shows that H' has a term bu_1^p for which, if the order of b is denoted by s, we have[73]

$$s = \sigma - \rho_1 p. \tag{103}$$

For R as at present, (66) vanishes for $w_0 = \varphi_1$ if $l_1 > 0$ or $l_2 > 0$. The vanishing also occurs for $l_1 = l_2 = 0$ if $l_0 < p$.[74] Using any l_0, l_1, l_2 for which (66) vanishes, let s_l be the order of the coefficient in (60) of $u_1^{l_0} u_{1x}^{l_1} u_{1y}^{l_2}$, assumed to be present in (60). By (70),

$$s_l > \sigma - \rho_1(l_0 + l_1 + l_2). \tag{104}$$

Our conclusion from this is that (p, s) is the lowest point of abscissa p plotted for H' and that (p, s) is associated only with the term bu_1^p.

From (103), we see that the line joining $(0, \sigma)$ and (p, s) has slope $-\rho_1$. From (104) and the remarks at the head of the preceding paragraph, it follows that any point of abscissa less than p, plotted for H', lies above this line.

Let now $l_0 + l_1 + l_2 > p$. We have by (69), (70),

$$s_l \geqq \sigma - \rho_1(l_0 + l_1 + l_2). \tag{105}$$

Thus every plotted point of abscissa greater than p lies on the line through (p, s) of slope $-\rho_1$, or above that line.

The above considerations make it geometrically obvious that if $\mathfrak{Q}'$ has sides of slope less than $-\rho_1$, the right extremity of the rightmost such side is (p, s).

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[72] The fact that σ' has no meaning of $u_1 = 0$ annuls H' does not interfere with our present work.

[73] σ is as in (57).

[74] This detail will be used somewhat further on.

50. There will fail to exist sides of slope less than $-\rho_1$ when, and only when, p is the least abscissa of the plotted points. In that case, $u_1 = 0$ annuls H' and

$$\frac{\partial H'}{\partial u_1}, \cdots, \frac{\partial^{p-1} H'}{\partial u_1^{p-1}} \tag{106}$$

vanish for $u_1 = 0$, while $\partial^p H'/\partial u_1^p$ does not. One sees, in that case that $u = \varphi_1 y^{\rho_1}$ is a y-solution of H of multiplicity p.

51. We study $\mathfrak{Q}'$ to the right of (p, s).

A term of H in

$$y^\mu u^\beta u_x^\gamma u_y^\delta, \tag{107}$$

associated with the point $(\beta + \gamma + \delta, \mu - \delta)$, contributes to H' terms coming from

$$y^\mu (u_1 + \varphi_1 y^{\rho_1})^\beta (u_{1x} + \varphi_{11} y^{\rho_1})^\gamma (u_{1y} + \rho_1 \varphi_1 y^{\rho_1 - 1})^\delta. \tag{108}$$

The terms coming from (108) will involve certain

$$y^{\mu + \rho_1(p_1 + p_2 + p_3) - p_3} u_1^{\beta - p_1} u_{1x}^{\gamma - p_2} u_{1y}^{\delta - p_3} \tag{109}$$

with $0 \leqq p_1 \leqq \beta$, etc. With each expression (109) is associated the point

$$(\beta + \gamma + \delta - (p_1 + p_2 + p_3), \quad \mu + \rho_1(p_1 + p_2 + p_3) - \delta). \tag{110}$$

One of the points (110) is $(\beta + \gamma + \delta, \mu - \delta)$. The others lie on a line sloping upward from that point, with slope $-\rho_1$.

This shows, firstly, that the right extremity of l, call it h, is a point plotted for H'. Also, every plotted point on $\mathfrak{Q}$ to the right of h is a point plotted for H'. If, then, we can prove that h is a vertex for $\mathfrak{Q}'$, it will follow that $\mathfrak{Q}'$ coincides with $\mathfrak{Q}$ to the right of h.

The line joining h to $(0, \sigma)$ has slope $-\rho_1$. Thus, if the abscissa q of h exceeds p, the line joining (p, s) to h has slope $-\rho_1$. What precedes shows that h is the lowest point of abscissa q plotted for H'. Hence, if $q > p$, $\mathfrak{Q}'$ has a side of slope $-\rho_1$ which starts at (p, s) and terminates at h. This side can have no b-points, for (105) is an inequality if one of l_1, l_2 is not zero. To the right of h, $\mathfrak{Q}$ and $\mathfrak{Q}'$ coincide.

If $q = p$, h is (p, s) and, as above, $\mathfrak{Q}$ and $\mathfrak{Q}'$ coincide to the right of h.

In any case, such b-points as $\mathfrak{Q}'$ may have lie on sides of slope less than $-\rho_1$.

52. Suppose that $\mathfrak{Q}'$ has sides of slope less than $-\rho_1$. Let us imagine that there are b-points on such sides,[75] and let l', of slope $-\rho_2 < -\rho_1$, be the leftmost side containing a b-point. Letting θ denote the intercept of l' on the axis of ordinates, we form a relation, analogous to (81),

$$H'(x, y, w_0 y^{\rho_2}, \cdots) = y^\theta R' + y^{\tau'} S' \tag{112}$$

[75] It will be seen later that this situation can be realized.

with R' involving w_1. We determine φ_2 as any solution of

$$R'(x, \varphi_2, \varphi_{21}, \rho_2\varphi_2) = 0, \tag{113}$$

which has an area of analyticity in common with φ_1. We can find an infinite number of solutions φ_2 with a common area of analyticity.[76] We then continue as in §§40–42.[77] For every φ_k with $k > 2$, we make the stipulation described in connection with (82). We obtain in this way an infinite set of y-solutions of H of the type described in §26. Obvious considerations relative to the analytic continuability of φ_1 show that we can find an area $\mathfrak{A}_2$ as in §26, contained in any area $\mathfrak{A}_1$. Thus, when $\mathfrak{Q}'$ has b-points, the y-solution number of F is ∞.

53. Suppose now that $\mathfrak{Q}'$ has no b-points. Let $p_1 \leqq p$ be the least abscissa of the points plotted for H'. If $p_1 > 0$, we see, as in §50, that $u = \varphi_1 y^{\rho_1}$ is a y-solution of H of multiplicity p_1.

Suppose that $p_1 < p$. We have just seen that $\varphi_1 y^{\rho_1}$ may be a y-solution of H. For any other y-solution with first term $\varphi_1 y^{\rho_1}$,

$$u_1 = \varphi_2 y^{\rho_2} + \cdots$$

must annul H'.[78] Proceeding as in §47, we find that ρ_2 must be the negative of the slope of a side of $\mathfrak{Q}'$. Also, ρ_2 having been selected, φ_2 satisfies an algebraic equation, analogous to (102) and of degree not greater than p,

$$R'(x, \varphi_2) = 0. \tag{114}$$

54. We are going to complete the proof of Theorem II.

Employing the hypothesis of Theorem II, we have, in §44, $n = \zeta_r \geqq 1$. Let t_1 stand for ζ_1 in (98).

If $t_1 = n$, it follows from §§45, 46, that $u = 0$ is the only y-solution of H and has multiplicity n. For this case, then, Theorem II holds. In what follows, we assume that $t_1 < n$.

We shall call p, defined as in §49, the *multiplicity* of φ_1.

By §45, H will have t_1 (possibly 0) y-solutions[79] $u = 0$. There will perhaps be certain possibilities $\varphi_1 y^{\rho_1}$ for first terms of other y-solutions.[80] The sum of t_1 and of the multiplicities of the φ_1 is n.

For each $\varphi_1 y^{\rho_1}$, we find an H' as in §49. If some H' has a b-point, the y-solution number of F is ∞. If no b-points are met, we proceed with each H' as in §53. We find that H has a certain number, say $t_2 \geqq t_1$[81] of y-solutions

[76] For the proof that φ_{21} appears effectively in (113), see §38.

[77] For H'', etc., we use the first sides of the corresponding polygons.

[78] We understand that $\varphi_2 \neq 0$.

[79] The language is clear.

[80] We are dealing with various ρ_1 as well as various φ_1. We take the φ_1 with a common area of analyticity.

[81] As above, multiplicities respected.

$u = 0$ or $u = \varphi_1 y^{\rho_1}$ and also, perhaps, a certain number of possibilities $\varphi_1 y^{\rho_1} + \varphi_2 y^{\rho_2}$ for the beginnings of other solutions. The sum of t_2 and of the multiplicities[82] of the φ_2 is n.

At the third step,[83] we form an H'' for each $\varphi_1 y^{\rho_1} + \varphi_2 y^{\rho_2}$. We continue in this manner. There are two ways in which our process, having been carried through k steps, may terminate at the $(k + 1)^{\text{th}}$ step. Firstly, we may meet an $H^{(k)}$ with a b-point.[84] In that case, the y-solution number of F is ∞.[85] Secondly, it may be that no $H^{(k)}$ has a side of slope less than the $-\rho_k$ associated with that $H^{(k)}$. In that case H will have precisely n y-solutions of the types $u = 0$ or

$$u = \varphi_1 y^{\rho_1} + \cdots + \varphi_h y^{\rho_h} \qquad (h \leqq k), \tag{115}$$

in harmony with Theorem II.[86]

Let us assume that the process does not terminate in a finite number of steps. Then, from some step on, no new finite solutions appear. That is, if k is large, we will, in the first k steps, have isolated a fixed number t of y-solutions composed of a finite number of terms, and there may be in addition a finite number of possibilities

$$\varphi_1 y^{\rho_1} + \cdots + \varphi_k y^{\rho_k} \tag{116}$$

for the beginnings of y-solutions with an infinite number of terms. The sum of t and of the multiplicities of the φ_k, for every large k, is n. After the finite y-solutions have been isolated, the number of distinct expressions (116) cannot decrease as k increases. Thus, after a certain step, there will be a fixed number of distinct expressions (116) and when $H^{(k)}$ is formed for any of these expressions, we get a φ_{k+1} with the same multiplicity as φ_k. For k large, $H^{(k)}$ does not vanish for $u_k = 0$ and its polygon has just one side of slope less than $-\rho_k$. We are thus carrying out a process entirely similar to that in §§34–42, and forming a certain number of y-solutions of H,

$$\varphi_1 y^{\rho_1} + \cdots + \varphi_k y^{\rho_k} + \cdots \tag{117}$$

each consisting of an infinite number of terms.

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[82] Definition as for φ_1.

[83] The details are as for the second step.

[84] Such b-points will lie on sides of slope less than $-\rho_k$. Of course, ρ_k may be different for different $H^{(k)}$.

[85] The matter of the areas $\mathfrak{A}_2$ in §26 is handled as follows. Consider any set of φ_j, $j = 1, \cdots, k$. We know thus far that the φ_j are analytic in some definite area. Every φ_j is determined by an algebraic equation. The coefficients in this equation are polynomials in $\varphi_1, \cdots, \varphi_{j-1}$, and their first derivatives, with functions in the underlying field for coefficients. By an induction, we can prove that φ_j satisfies an algebraic equation with coefficients in the field. It follows that the φ_j can be continued along any path in $\mathfrak{A}$ which contains no point of a certain point set which has no limit point in $\mathfrak{A}$. This is enough to give the areas $\mathfrak{A}_2$.

[86] For any y-solution (115), the φ_j can be continued over any simply connected area which contains no point of the point set mentioned in the preceding footnote. For $\mathfrak{S}$ in §25, we take the logical sum of these sets for the various y-solutions (115).

Let us study the φ_k in any y-solution (117) from the point of view of analytic continuation. We work with a particular y-solution (117) and, in speaking of expressions $H^{(k)}$, understand them to be associated with that y-solution.

We denote by p the common multiplicity of the φ_k with k large in (117). Let, for k large, $\sigma^{(k)}$ be the order of $H^{(k)}(0)$. We have a relation, analogous to (65),

$$H^{(k)}(u_k) = y^{\sigma^{(k-1)}} R^{(k-1)}(x, \varphi_k + y^{-\rho_k} u_k) + y^{\tau^{(k-1)}} S^{(k-1)}. \tag{118}$$

The equation

$$R^{(k-1)}(x, \varphi_k) = 0$$

is of degree p and its roots are all equal. If the coefficients of φ_k^p and φ_k^{p-1} are respectively a_k and b_k, we have

$$\varphi_k = -\frac{b_k}{pa_k}. \tag{119}$$

Here a_k and b_k are polynomials in $\varphi_1, \cdots, \varphi_{k-1}$ and their first derivatives, with functions in the underlying field for coefficients.

We wish to show that a_k does not depend on k. By (118), $H^{(k)}$ has a term $a_k y^s u_k^p$ with

$$s = \sigma^{(k-1)} - p\rho_k. \tag{120}$$

The point (p, s) is the right extremity of the first side of the polygon of $H^{(k)}$. By §51, (p, s) is also the right extremity of the first side of the polygon of $H^{(k+1)}$, and the details of that section show that $H^{(k+1)}$ has a term $a_k y^s u_{k+1}^p$. Thus $a_{k+1} = a_k$.

We represent the equal functions a_k by d. We see now, by (119), that there is a positive integer h such that for $k > h$, φ_k is rational in $\varphi_1, \cdots, \varphi_h$ and their derivatives of various orders, the denominator of φ_k being a power of d. Now $\varphi_1, \cdots, \varphi_h$ and their derivatives, as was seen above, can be continued along any path which avoids the points of a certain set which has no limit point in $\mathfrak{A}$. The same is true of $1/d$. It follows that all φ_k in (117) can be continued along any path which avoids a certain set with no limit point in $\mathfrak{A}$. The φ_k can be continued, as uniform analytic functions, over any simply connected area containing no point of the set.

Let us think of the t finite y-solutions and of the y-solutions (117) as having been constructed with their coefficients all analytic in some definite area. If these coefficients are continued over some area $\mathfrak{A}_1$, we get a set of y-solutions with one of which any y-solution of F with coefficients analytic at a point in $\mathfrak{A}_1$ must coincide.

To complete the proof of Theorem II, we take a y-solution (117) and show that p, the common multiplicity of the φ_k with k large, is a multiplicity for the y-solution. The fact that the equation

$$R^{(k)}(x, w_0) = 0,$$

with k large, has p roots all equal to φ_{k+1} in (117), shows that the first side of the polygon for $H^{(k)}$ has on it points plotted for $H^{(k)}$ of abscissas $0, \cdots, p$, associated with terms in u_k^l, $l = 0, \cdots, p$. Thus the coefficient of u_k^l in $H^{(k)}$, for $l = 0, \cdots, p$, has an order equal to

$$\sigma^{(k)} - l\rho_{k+1}.$$

This means that if u_k is replaced by 0 in

$$\frac{\partial^l H^{(k)}(u_k)}{\partial u_k^l},$$

$l = 1, \cdots, p$, we get an expression in x and y in which the least exponent of y is $\sigma^{(k)} - l\rho_{k+1}$.

We have seen that s in (120), which is the ordinate of the right extremity of the first side of the polygon of $H^{(k)}$, is the same for all large k. We have thus

$$s = \sigma^{(k)} - p\rho_{k+1}.$$

Thus

$$\sigma^{(k)} - l\rho_{k+1} = s + (p - l)\rho_{k+1}. \tag{121}$$

The expression in x and y mentioned above is obtained again if we replace u in $\partial^l H(u)/\partial u^l$ by

$$\varphi_1 y^{\rho_1} + \cdots + \varphi_k y^{\rho_k}.$$

Now the second member of (121) is large when k is large, for $l = 1, \cdots, p - 1$, but has a fixed value s for $l = p$. This shows that (117) annuls the first $p - 1$ derivatives of H with respect to u, but not the p^{th} derivative. Thus the y-solution (117) is of multiplicity p.

Theorem II is thus proved. We proceed to show how this theorem can be used to test for the presence of $y = 0$ in the general solution of F. Following the indications in §22, we shall develop a method for determining whether F has a y-solution which does not annul any B_i which may be present in (39).

Multiplicities and Vanishing Derivatives

55. Let there be B_i in (39) and let B represent some one of them.

A series (40) which either vanishes identically or else has $\rho_1 \geqq 1$, will, if it annuls B, be called a *y-solution* of B. Every y-solution of B is a y-solution of F.

Let B have a y-solution $\bar{u}$. We shall show that certain partial derivatives of H are annulled by $\bar{u}$. Also we shall prove that $\bar{u}$ has a multiplicity and we shall determine that multiplicity.

According to §5, there is a $t \geqq 0$ such that, S being the separant of B, we have

$$S^t F = C_0 B^p + C_1 B^{p_1} B'^{q_1} + \cdots + C_r B^{p_r} B'^{q_r} \tag{122}$$

with B' the derivative of B, where the C are forms of order one at most, not divisible by B, and where

$$p > 0; \qquad p < p_i + q_i, \qquad i = 1, \cdots, r.$$

We are going to prove that $\bar{u}$ has the multiplicity p. Also, we shall prove that

$$\frac{\partial^{l_0+l_1+l_2} H}{\partial u^{l_0} \partial u_x^{l_1} \partial u_y^{l_2}}$$

is annulled by $\bar{u}$ for $l_0 + l_1 + l_2 \leqq p$ if at least one of l_1, l_2 is not zero.

We have

$$S^t H = C_0 B^p + C_1 B^{p_1} B'^{q_1} + \cdots \tag{123}$$

where, in S, B, B' and the C, the substitution

$$y_1 = u, \qquad y_2 = uu_y + u_x \tag{124}$$

is supposed to be made.

Then H, B and B' are annulled by $\bar{u}$, but S and the C are not.[87] In the identity (123), we put

$$u = \bar{u} + w_0, \qquad u_x = \bar{u}_x + w_1, \qquad u_y = \bar{u}_y + w_2 \tag{125}$$

where the w are indeterminates. Because $\bar{u}$ does not annul S or any C, we have, for (125),

$$\begin{aligned} S &= a_0 + a_1 w_0 + a_2 w_0^2 + \cdots, \\ C_i &= b_{0i} + b_{1i} w_0 + b_{2i} w_0^2 + \cdots, \end{aligned} \tag{126}$$

where the a and b are expressions in x and y, a_0 and the b_{0i} being distinct from zero. Also, for (125),

$$B = a_0 w_0 + d_2 w_0^2 + d_3 w_0^3 + \cdots$$

with a_0 as in (126); and

$$B' = f w_0 + g w_1 + h w_2 + k w_0^2 + \cdots.$$

Thus, for (125), the second member of (123) becomes a polynomial in the w_i, effectively containing a term in w_0^p but no term in w_0^j with $j < p$. Furthermore, no power product of degree less than $p + 1$, in which w_1 or w_2 is present, appears in the polynomial.

Because $a_0 \neq 0$, (123) shows that H also becomes, for (125), a polynomial with the properties just described. This implies the results stated above with respect to the partial derivatives of H and the multiplicity of $\bar{u}$.

Final Criteria

56. Let F be as in the hypothesis of Theorem II. Let one or more B_i be present in (39), and let $y = 0$ belong to the general solutions of certain of them, say to the general solutions of[88]

$$B_1, \cdots, B_t. \tag{127}$$

[87] S and the C, as forms in y, are not divisible by B, and B is algebraically irreducible.

[88] This assumption holds through §65.

Our present object is to determine whether F has y-solutions which annul none of[89] $B_1, \cdots, B_t$.

Let each B_i, $i = 1, \cdots, t$, be considered as a polynomial in y, y_1 and let q_i be the highest exponent of y_1 in the terms of lowest degree in B_i. One can see, from the Newton polygon for B_i, that, for x in a suitable area, B_i has q_i distinct y-solutions with coefficients analytic throughout the area. Let p_i be the value of p in (123) for $B = B_i$. Let

$$m = p_1q_1 + \cdots + p_tq_t.$$

We compare m with n as in Theorem II.

Suppose that $n > m$.[90] Then, whether the y-solution number of F is n or ∞, F must have a y-solution which annuls no B_i in (127).

Suppose that $n < m$. Then n cannot be the y-solution number of F, so that the y-solution number of F is ∞ and F has y-solutions which annul no B_i in (127).

Now let $n = m$. In the sections which follow, it will be shown how to determine for this case, by a finite number of operations, whether the y-solution number of F is n or ∞. If the y-solution number is ∞, F has y-solutions which annul no B_i. If the y-solution number is n, we know that any y-solution of F with coefficients analytic at a point not contained in $\mathfrak{E}$ of §25 annuls some B_i. Now the work in §54 connected with (119) shows that when F has a finite y-solution number, the coefficients in any y-solution, if they are all analytic at a point a, are analytic together in some neighborhood of a. Thus when the y-solution number is n, every y-solution of F annuls some B_i.

57. Assuming that $m = n$, we undertake to decide whether the y-solution number of F is n or ∞.

We start by showing how to determine a positive integer k such that, given two y-solutions of the same or different B in (127), all coefficients in both y-solutions having a common area of analyticity, the two y-solutions do not coincide through their first k terms.[91]

Let $A = B_1B_2 \cdots B_t$. Let A be of degree q in y_1. Then, for x in a suitable area, A is annulled by precisely q distinct power series

$$y_1 = \varphi_1 y^{\rho_1} + \varphi_2 y^{\rho_2} + \cdots \tag{128}$$

with increasing rational ρ_i, for some of which ρ_1 may be less than 1. One of these series may be identically zero.

[89] The remaining B have no y-solutions.

[90] Examples with $n > m$, $n = m$, $n < m$, will be given in §89.

[91] Two distinct y-solutions of which one is $y_1 = 0$ will be considered not to coincide through their *first* terms. If a y-solution consists of a finite number, say g, of non-vanishing terms, those terms are supposed to be the first g terms and we imagine an infinite number of vanishing terms to follow. Otherwise, we allow only non-vanishing terms. When we refer to k as above in a case in which there is only one B_i, with only one y-solution, k will be understood to be any positive integer.

The product ξ of the $q(q-1)$ differences of pairs of solutions (128) is a rational function of y which it is possible to calculate by the theory of symmetric functions.

The least possible value s of ρ_1 in (128), for the solutions (128) distinct from zero, can be found by inspection from the Newton polygon of A. The difference of any two solutions (128) is thus a series

$$\zeta_1 y^{\sigma_1} + \zeta_2 y^{\sigma_2} + \cdots \tag{129}$$

with $\sigma_1 \geqq s$. If, then, the expansion of ξ in ascending powers of y begins with the g^{th} power of y, we have, for any σ_1 in (129),

$$\sigma_1 + s[q(q-1) - 1] \leqq g$$

or

$$\sigma_1 \leqq g - s(q^2 - q - 1). \tag{130}$$

We denote the second member of (130) by v. Then no two of the y-solutions of the B coincide through a power of y as high as the v^{th} power. Let w be the greatest of the degrees of the B in y_1. Then the common denominator of the ρ_i in any y-solution may be taken not greater than w.[92] It suffices then to take

$$k \geqq (v-1)w + 1.$$

58. Let k be any integer such that no two y-solutions of the B in (127) coincide through their first k terms.

We return to the process described in §§31–54 for seeking y-solutions (not all) of H. Let g be any positive integer not greater than k. If the process terminates at the g^{th} step,[93] it must be that either we have encountered an $H^{(g-1)}$ with b-points, or that no $H^{(g-1)}$ has a side of slope less than the corresponding[94] $-\rho_{g-1}$. The manner of termination would indicate whether the y-solution number of F is n or ∞.

To test whether the process terminates at the k^{th} step, or at an earlier step, requires a finite number of rational operations, differentiations and resolutions of polynomials into irreducible factors. To be sure, we may have to solve algebraic equations like (102), but all that is necessary for handling a solution of such an equation is the knowledge of an irreducible equation which such a solution satisfies. It is then possible to calculate with the solution and its derivative by the methods of abstract algebra.

59. Let us suppose that the process does not terminate at the k^{th} step or at an earlier step. We shall prove, in §§60–65, that the y-solution number of F is n.

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[92] $B_i = 0$ defines y_1 as a function of x, y, of at most w branches.

[93] See §54.

[94] For $g = 1$, this means that either $H^0 = H$ has b-points or $\mathfrak{Q}$ consists of a single point.

60. The non-termination means that we have met certain $H^{(k-1)}$ with sides of slopes less than the associated $-\rho_{k-1}$ and with no b-points.[95] There will have been isolated a certain number t_k of y-solutions which either are zero or possess at most $k - 1$ terms. The sum of t_k and the multiplicities of the φ_k which the $H^{(k-1)}$ yield is n.

Thus $t_k < n = m$. Then the B_i must have y-solutions with at least k terms[96] and the ρ_k of the k^{th} terms must be negatives of slopes of sides of the $H^{(k-1)}$.

Let

$$\varphi_1 y^{\rho_1} + \cdots + \varphi_k y^{\rho_k} + \cdots, \tag{131}$$

consisting of at least k terms be a y-solution of some B_i, say B_1. Let (131) have multiplicity p for H. Then ρ_k in (131) is the negative of the slope of a side of some $H^{(k-1)}$, which, for the substitution

$$u_{k-1} = \varphi_k y^{\rho_k} + u_k$$

yields an $H^{(k)}$ which is annulled by[97]

$$\varphi_{k+1} y^{\rho_{k+1}} + \cdots. \tag{132}$$

61. If $H^{(k)}$ has sides of slope less than $-\rho_k$, let q be the abscissa of the right extremity of the rightmost such side. Otherwise, let q be the least abscissa of the points plotted for $H^{(k)}$. As was seen in §§49, 50, q is the multiplicity of φ_k. We shall prove that $q = p$.

In the above, $H^{(k-1)}$, $H^{(k)}$, (131), p and q were fixed entities. In the paragraph which follows, but not beyond, we consider various $H^{(k-1)}$ etc.

The sum of t_k and of the various q for all $H^{(k)}$ obtained from all $H^{(k-1)}$ is n.[98] The sum of t_k and of the p for all possible (131) for the various B_i is at least $m = n$. Two distinct (131) associated with the same $H^{(k-1)}$ differ either in ρ_k or in φ_k and hence are associated with distinct $H^{(k)}$.

It follows that, for the particular p and q which we are using, it will suffice to prove that $q \geqq p$.[99]

62. Suppose that $q < p$. By §49, the point on the polygon of $H^{(k)}$ of which q is the abscissa is associated with a single term, namely a term in u_k^q. We denote this point by h. If h is joined to any point plotted for $H^{(k)}$ of abscissa exceeding q, we get a slope not less than $-\rho_k$.

Let K represent $\partial^q H^{(k)}/\partial u_k^q$. Because $q < p$, K is annulled by (132), as $H^{(k)}$ is.

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[95] If $k = 1$, this means that $\mathfrak{Q}$ has a side of slope not greater than -1 and has no b-points.

[96] If a B_i has a y-solution which is zero or has fewer than k terms, that y-solution must be among the t_k y-solutions mentioned above.

[97] (132) may be zero.

[98] Here we are including even those $H^{(k)}$ which may not be related as above to a (131).

[99] When this is shown, we shall know also that every one of the t_k y-solutions mentioned above annuls some B_i.

Let points be plotted for K by the method used for the various $H^{(i)}$. A point for $H^{(k)}$, associated with a term involving u_k to the q^{th} power at least, yields a point for K by a translation to the left of q units. All other points for $H^{(k)}$ disappear. Thus h produces, for K, a point h' on the axis of ordinates which is the lowest point for K on that axis. Also, if h' is joined to any point for K of positive abscissa, we get a slope not less than $-\rho_k$. Because K has a point on the axis of ordinates, K is not annulled by (132) if (132) is zero. Because of the slope situation just described, K cannot be annulled by (132) with $\rho_{k+1} > \rho_k, \varphi_{k+1} \neq 0$. (§§46, 47.)

63. Thus, $q = p$. If p is the least abscissa for the points of $H^{(k)}$, (132) is zero and $H^{(k)}$ can be discarded in the search for y-solutions of H. Suppose conversely that (132) is zero. We shall prove that p is the least abscissa for $H^{(k)}$.

Otherwise $H^{(k)}$ would have a term in

$$u_k^{l_0} \, u_{kx}^{l_1} \, u_{ky}^{k_2}$$

with $l_0 + l_1 + l_2 < p$. The coefficient of that term would be

$$\frac{\partial^{l_0+l_1+l_2} H^{(k)}}{\partial u_k^{l_0} \, \partial u_{kx}^{l_1} \, \partial u_{ky}^{l_2}} \tag{133}$$

with u_k, u_{kx}, u_{ky} replaced by zero. However, §55 shows that (133) will vanish for these replacements, because the corresponding derivative of H vanishes for (131).

Thus, for (132) to be zero, it is necessary and sufficient that p be the least abscissa for $H^{(k)}$.

64. We suppose that (132) is not zero, so that $H^{(k)}$ has at least one side of slope less than $-\rho_k$. We shall prove that $H^{(k)}$ has a single such side and that this side has slope $-\rho_{k+1}$ (as in (132)) and has a horizontal projection of length p. That is, the side joins the point h to a point on the axis of ordinates. It will be seen also that this side has no b-points.

We begin by showing that the rightmost side of slope less than $-\rho_k$, call it l, has a slope not less than $-\rho_{k+1}$. Let the contrary be assumed. Then a line through h of slope $-\rho_{k+1}$ will have every point other than h plotted for $H^{(k)}$ lying above it. It follows that (132) cannot annul $H^{(k)}$.

We prove now that l contains no b-point. Suppose that l has a b-point and let the rightmost b-point on l be designated by h_1. The abscissa of h_1 is less than p. Let one of the terms associated with h_1 be a term in

$$u_k^{l_0} \, u_{kx}^{l_1} \, u_{ky}^{l_2} \tag{134}$$

with l_1 and l_2 not both zero. Let

$$K = \frac{\partial^{l_0+l_1+l_2} H^{(k)}}{\partial u_k^{l_0} \, \partial u_{kx}^{l_1} \, \partial u_{ky}^{l_2}}.$$

According to §55, K is annulled by (132). We plot points for K in the customary manner. Then a point for $H^{(k)}$ yields a point $l_0 + l_1 + l_2$ units to the left and l_2 units higher up, or yields no point for K, according as the point for $H^{(k)}$ does or does not have a term associated with it which is divisible by (134). In particular, h_1 yields a point h_2 on the axis of ordinates (the lowest such point for K) and no point on l to the right of h_1 yields a point for K. On this basis, we see that if h_2 is joined to any point for K of positive abscissa, we obtain a slope greater than the slope of l, that is, greater than $-\rho_{k+1}$. It follows that (132) cannot annul K.

Thus, l has only a-points. We show that the slope of l is $-\rho_{k+1}$. Suppose that the slope is greater. Let h_1, of abscissa $p_1 < p$, be the left extremity of l. Let

$$K = \frac{\partial^{p_1} H^{(k)}}{\partial u_k^{p_1}}.$$

We see that K has at least one point on the axis of ordinates and that when the lowest such point, obtained from h_1, is joined to a point for K of positive abscissa, we get a slope greater than $-\rho_{k+1}$. It follows that (132) does not annul K, a contradiction.

We prove finally that the horizontal projection of l is of length p. Otherwise the multiplicity of φ_{k+1} in (132) would be less than p and $H^{(k+1)}$ would have a point h_1, analogous to h, of abscissa $p_1 < p$, which is either the rightmost extremity of sides of slope less than $-\rho_{k+1}$ or else a least abscissa. Then h_1 would be associated with a single term, namely a term in $u_{k+1}^{p_1}$, and $\partial^{p_1} H^{(k+1)}/\partial u_{k+1}^{p_1}$ could not vanish for

$$u_{k+1} = \varphi_{k+2} y^{\rho_{k+2}} + \cdots,$$

a contradiction.

65. What we have shown, in the preceding sections, is that if the process does not terminate in k or fewer steps, then, if the process terminates at the $(k+1)^{\text{th}}$ step, H has precisely n y-solutions (for a suitable area), each consisting of a finite number of terms. Now $k + 1$ is an integer of the type described in §57. It follows that the process cannot terminate by the appearance of b-points, so that the y-solution number of F is n.

A Special Case

66. We consider a case, of frequent occurrence, in which the y-solution number of F is n rather than ∞.

Let F be as in the hypothesis of Theorem II. We make no assumption as to the existence of forms B as in (127) and, if such B exist, nothing is assumed relative to m.

Suppose that H has no b-points and that the various φ_1 found for the first terms of y-solutions all have multiplicity unity.[100] The number of y-solutions $u = 0$, that is, the abscissa of the leftmost point of H, is of no importance.

We shall prove that the y-solution number of F is n. It is seen under the assumption made, from §§49, 50, that every $H^{(k)}$ encountered, $k \geqq 1$, has a term in the first power of u_k whose associated point, an a-point, is either the leftmost point on the polygon of $H^{(k)}$ or else is the right extremity of the only side of slope less than $-\rho_k$. Because the a-point mentioned has unity for abscissa, $H^{(k)}$ can have no b-points. This proves our statement.

Series in the General Solution

67. Let F be an algebraically irreducible form in y, of the second order. We denote the general solution of F by $\mathfrak{M}$.

If F has y-solutions which annul every form[101] holding $\mathfrak{M}$, such y-solutions will be said to *be in* $\mathfrak{M}$. Suppose that there is a y-solution in $\mathfrak{M}$. Then $y = 0$ is in $\mathfrak{M}$, for a form containing a term in x alone cannot be annulled by a y-solution.

In the sections which follow, we shall establish the converse of this fact, namely that if $y = 0$ belongs to $\mathfrak{M}$, there is a y-solution in $\mathfrak{M}$. More definitely, if $y = 0$ is in $\mathfrak{M}$, there is a set of points, dense in the area in which the coefficients in F are analytic, such that, given any point a of the set, there exists a y-solution in $\mathfrak{M}$ whose coefficients are analytic at a.

Let us prove that if there are B in (39), no y-solution in $\mathfrak{M}$ can annul any B. Consider any B_i and, with it, D_i of §20, which is annulled by all solutions in $\mathfrak{M}$ which annul B_i. D_i is of order zero. If $G_1, \cdots, G_g$ is a finite set of forms whose manifold is $\mathfrak{M}$, some power of D_i is a linear combination of $B_i, G_1, \cdots, G_g$ and the derivatives of those forms, with forms for coefficients. Any y-solution in $\mathfrak{M}$ which annuls B_i would thus annul D_i. As D_i does not involve y_1, D_i cannot vanish for a y-solution.

We prove now that a y-solution of F which annuls no B in (39) is in $\mathfrak{M}$.[102] Let R be the product of the B_i in (39) and T the product of the C_i. Let $U = RT$. Then U and R are annulled by the same y-solutions of F, namely the y-solutions of the B_i. If Q is any form which holds $\mathfrak{M}$, UQ holds F. It follows, as above, that every y-solution of F annuls UQ. Hence, a y-solution of F which annuls no B_i annuls Q and is in $\mathfrak{M}$.

Thus, after it has been proved that $\mathfrak{M}$ contains y-solutions if it contains $y = 0$, the problem of determining whether $y = 0$ is in $\mathfrak{M}$ will have been reduced to the problem of determining whether or not F has y-solutions which do not annul any B which may exist in (39). This question is covered by the preceding sections.

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[100] For an example, see §89.

[101] The successive derivatives of the y-solutions are calculated formally as in §21.

[102] If there are no B, every y-solution is in the general solution.

Indecomposable Systems

68. In *A. D. E.*, §64, we proved the following theorem:

Let Σ be an indecomposable system of simple forms in $y_1, \cdots, y_n$.[103] *Let B be any simple form which does not hold Σ. Given any solution of Σ, analytic in an area $\mathfrak{A}_1$, there is an area $\mathfrak{A}'$, contained in $\mathfrak{A}_1$, in which the given solution can be approximated uniformly, with arbitrary closeness, by solutions of Σ for which B is distinct from 0 throughout $\mathfrak{A}'$.*

We are going to establish here the following stronger result:

Let Σ be an indecomposable system of simple forms in $y_1, \cdots, y_n$ and B any form which does not hold Σ. Let $y_i = \zeta_i$, $i = 1, \cdots, n$, be any solution of Σ, and $\mathfrak{A}_1$ an area in which this solution is analytic. There exist n functions $\varphi_i(x, h)$, $i = 1, \cdots, n$, of x and a parameter h, analytic for h small and for x in some area contained in $\mathfrak{A}_1$, such that $\varphi_i(x, 0) = \zeta_i$, $i = 1, \cdots, n$, and such that the φ_i annul every form in Σ, but not B, identically in x and h, when substituted for the y_i.

To prove this we take up again the considerations of §§56–61 of *A. D. E.*, and, beginning with §§62, 63, introduce some modifications.

Let $\xi_1; \cdots, \xi_n$ be any solution of Ψ, analytic in some area $\mathfrak{A}_1$. Let H be any non-vanishing simple form in $z_1, \cdots, z_q$. We shall show the existence of n functions $\varphi_i(x, h)$, $i = 1, \cdots, n$, analytic for h small and for x in some area in $\mathfrak{A}_1$, such that $\varphi_i(x, 0) = \xi_i$ and such that every form in Σ_1 vanishes identically in x and h for $z_i = \varphi_i(x, h)$, while H does not.

If H does not vanish for $z_i = \xi_i$, $i = 1, \cdots, n$, we take $\varphi_i = \xi_i$. In what follows, we assume that H vanishes for $z_i = \xi_i$. We proceed as in *A. D. E.*, §63. Starting with the equations (43) of that section, which we rewrite here

$$z_i = \xi_i + b_i h, \qquad i = 1, \cdots, q, \tag{135}$$

we observe that $R = 0$ will admit as solutions for w, when (135) holds, g distinct series of the type

$$\delta_1 h^{\rho_1} + \delta_2 h^{\rho_2} + \cdots,$$

convergent for $| h |$ small, where the ρ_i are non-negative rational numbers, with a common denominator, which increase with i, and where the δ are functions of x.[104]

These g expressions for w give, by means of the equations (39) of *A. D. E.*, g distinct sets of power series for $z_{q+1}, \cdots, z_n$ which, together with the second members of (135) above, annul every form in Σ_1. This is because the power series furnish solutions of Σ_1 for $| h |$ small. Because each z_i, $i > q$, satisfies an equation with $z_1, \cdots, z_q$ in which the highest power of z_i has unity for coefficient

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[103] We use the letters n, B and H in this section as in the corresponding sections of *A. D. E.* No confusion with the notation of the present paper will result.

[104] Because the coefficient of w^g in R is free of $z_1, \cdots, z_q$, there can be no negative ρ_i.

(*A. D. E.* §57), the power series for $z_{q+1}, \cdots, z_n$ cannot contain negative powers of h. Let the g sets of power series be represented by

$$z_{q+1}^{(k)}, \cdots, z_n^{(k)}, \qquad k = 1, \cdots, g. \tag{136}$$

For $|h|$ small, if (135) holds, Z of *A. D. E.*, §60, will vanish if

$$v = v_k = u_1 z_{q+1}^{(k)} + \cdots + u_p z_n^{(k)}, \qquad k = 1, \cdots, g. \tag{137}$$

Since the g expressions v_k are distinct from one another, we have, representing by β the polynomial in v and the u_i which Z becomes when (135) holds,

$$\beta = (v - v_1) \cdots (v - v_k).$$

Let v_k with $h = 0$ be denoted by v_k'. Representing Z, for $z_i = \xi_i$, $i = 1, \cdots, q$, by γ, we have

$$\gamma = (v - v_1') \cdots (v - v_k').$$

As $\xi_1, \cdots, \xi_n$ is a solution of Ψ,

$$v - u_1\xi_{q+1} - \cdots - u_p\xi_n$$

must be a factor of γ. This shows that, for some k, (136) reduces to $\xi_{q+1}, \cdots, \xi_n$ for $h = 0$. Using the set (136) for this k, and replacing h by a suitable integral power of itself, we obtain, from (135) and (136), a set of $\varphi_i(x, h)$, analytic in x and in the new parameter h, which fulfill our requirements.

We continue as in *A. D. E.*, §64. Equation (51) of that section shows that

$$D^{\mu}C - N \tag{138}$$

with w replaced by $a_1 z_{q+1} + \cdots + a_p z_n$ is a form in Σ_1. Let H be formed as in *A. D. E.*, §64, (52). We determine $\varphi_i(x, h)$, $i = 1, \cdots, n$, as above, so as to reduce to the ξ_i for $h = 0$ and to annul every form in Σ_1 but not H. Then, as the φ_i annul R (with w replaced as above), they do not annul N. We see by (138) that the φ_i do not annul C. As the y_i are linear in the z_i, we have the desired modification of the result in *A. D. E.*

Approximate Series Solutions

69. We take F algebraically irreducible and of the second order. Let $y = 0$ belong to the general solution of F.

For any positive integer p, we consider

$$F, F_1, \cdots, F_p, \tag{139}$$

where F_j is the j^{th} derivative of F. Let S be the separant of F. Denoting y by y_0, we consider S and the forms in (139) as simple forms in $y_0, y_1, \cdots, y_{p+2}$. In the decomposition of (139) into essential prime systems, there is precisely one prime system—call it Λ—which is not held by S. Then $y_i = 0$, $i = 0, \cdots, p + 2$, is a solution of Λ. This is because a form G in Λ, if considered as a differential polynomial, vanishes for all normal solutions of F and hence for $y = 0$.

We are going to show the existence of certain functions $\psi_i(x, h)$, $i = 0, \cdots, p+2$, analytic for x in some area and for $|h|$ small, and reducing to 0 for $h = 0$, such that every form in Λ, but not S, vanishes identically in x and h for $y_i = \psi_i$, $i = 0, \cdots, p+2$. The function ψ_0 will not be identically zero and the ψ_i will be such that, when they are expanded in powers of h, the lowest power of h present in the set of them is present in ψ_0. That is, we shall have

$$\psi_i = \zeta_{ri}h^r + \cdots + \zeta_{ki}h^k + \cdots, \qquad (i = 0, \cdots, p+2) \tag{140}$$

with the ζ analytic in x, with r a positive integer and with $\zeta_{r0} \neq 0$.

For the proof, we put $y_i = y_0 u_i$, $i = 1, \cdots, p+2$. Then Λ goes over into a system Λ_1 in

$$y_0, u_1, \cdots, u_{p+2}. \tag{141}$$

Λ_1 has solutions with $y_0 \neq 0$, because Λ has. If G_1 and G_2 are forms in the unknowns (141) such that G_1G_2 vanishes for all solutions of Λ_1 with $y_0 \neq 0$, then G_1 vanishes for all such solutions or G_2 does. This is because Λ is indecomposable. Therefore the totality of forms which vanish for all solutions of Λ_1 with $y_0 \neq 0$ constitute a prime system Λ'. The manifold of Λ_1 consists of that of Λ' and of solutions with $y_0 = 0$.

We shall prove that Λ' has solutions with $y_0 = 0$. Let this be false. Let

$$G_1, \cdots, G_r \tag{142}$$

be a finite subset of Λ' with the same manifold as Λ'. Then the system $G_1, \cdots, G_r, y_0$ has no solutions. There exists thus a relation

$$Ky_0 + K_1G_1 + \cdots + K_rG_r = 1, \tag{143}$$

where the K are simple forms in $y_0, u_1, \cdots, u_{p+2}$. Then $1 - Ky_0$ is a form in Λ'. This means that

$$1 - y_0K\left(x; y_0, \frac{y_1}{y_0}, \cdots, \frac{y_{p+2}}{y_0}\right) \tag{144}$$

vanishes for all solutions of Λ with $y_0 \neq 0$. K as in (144) cannot be free of $y_1, \cdots, y_{p+2}$, for Λ is not held by a form in y_0 alone. Let K in (143) be of degree $g \geqq 1$ as a polynomial in $u_1, \cdots, u_{p+2}$. Then

$$y_0^gK\left(x; y_0, \frac{y_1}{y_0}, \cdots, \frac{y_{p+2}}{y_0}\right) = L(x; y_0, y_1, \cdots, y_{p+2})$$

where L is a simple form, each of whose terms is of degree at least g in $y_0, \cdots, y_{p+2}$. From (144) we see that

$$y_0^{g-1} - L,$$

which we denote by M, vanishes for all solutions of Λ with $y_0 \neq 0$. Because y_0 does not hold Λ, M does. Then some irreducible factor N of M holds Λ. The term y_0^{g-1} in M has a lower degree than any other term in M. Hence N has a

term in y_0 alone, which is of lower degree than any other term in N.[105] Let N be considered as a differential polynomial. By §5, $y = 0$ is an essential manifold in the manifold of N. But N holds the general solution of F so that $y = 0$ cannot be essential for N.

Thus Λ' has solutions with $y_0 = 0$. Let $(0, \bar{u}_1, \cdots, \bar{u}_{p+2})$ be such a solution. Now $y_0 S$, with $y_i = y_0 u_i$ in S for $i = 1, 2$, does not hold Λ'. By §68, there are $p + 3$ analytic functions $\varphi_i(x, h)$, $i = 0, \cdots, p + 2$, which reduce respectively to $0, \bar{u}_1, \cdots, \bar{u}_{p+2}$ for $h = 0$ and which annul all forms in Λ', without annulling $y_0 S$, when substituted for $y_0, u_1, \cdots, u_{p+2}$. The failure of $y_0 S$ to vanish implies that $\varphi_0(x, h)$ is not identically zero.

Let

$$\psi_0(x, h) = \varphi_0(x, h); \quad \psi_i(x, h) = \varphi_0(x, h)\varphi_i(x, h), \qquad i = 1, \cdots, p + 2.$$

Then the ψ reduce to 0 for $h = 0$ and annul all forms in Λ without annulling S. Let the ψ be expanded in powers of h. Then the lowest exponent in ψ_0 does not exceed the lowest exponent in any other ψ_i. The ψ are thus as described above.

70. Let a be a value of x such that:

I. The coefficients in F are analytic at a.

II. Every $\psi_i(x, h)$ is analytic for $x = a$ and for $|h|$ small.[106]

III. $\zeta_{r0}(a)$ as in (140) is not zero.

IV. S does not vanish identically in h for

$$x = a, \qquad y_i = \psi_i(a, h), \qquad i = 0, 1, 2.$$

We let $\alpha_i(h) = \psi_i(a, h)$, $i = 0, \cdots, p + 2$.

For $x = a$, F vanishes identically in h if we put $y_i = \alpha_i$, $i = 0, 1, 2$. Because of IV above, the equations $F_i = 0$, $i = 1, \cdots, p$, with $x = a$, $y_i = \alpha_i$, $i = 0, 1, 2$, determine $y_3, \cdots, y_{p+2}$ in succession as the α_i for $i > 2$.

Let F_j represent for $j > p$, as above for $j \leqq p$, the j^{th} derivative of F. Taking y_i equal to α_i for $i \leqq p + 2$, we can now solve the (algebraic) equations $F_j = 0$, $j = p + 3, p + 4, \cdots$, in succession for $y_{p+3}, y_{p+4}, \cdots$. For every $j > p + 2$, we find $y_j = \alpha_j(h)$ where α_j is either analytic, or else has a pole, for $h = 0$. Then, for $|h|$ small and distinct from zero,

$$\alpha_0(h) + \alpha_1(h)(x - a) + \cdots + \frac{\alpha_k(h)}{k!}(x - a)^k + \cdots \tag{145}$$

is a normal solution of the differential polynomial F.[107]

By III above, the lowest exponent of h in α_0 is r of (140). That in α_k for $0 < k \leqq p + 2$ is at least r. For $k > p + 2$, the lowest exponent may be less than r and may even be negative.

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[105] The sum of the terms of lowest degree in N must be a factor of y_0^{g-1}. Because N vanishes for $y_i = 0$, $i = 0, \cdots, p + 2$, N cannot have a term free of the y_i.

[106] We are dealing with a definite value of p and with a definite set of ψ_i as in §69.

[107] x is restricted to a neighborhood of a which may, for all we know, depend on h.

71. Let Φ be a finite system of differential polynomials, containing F, whose manifold is the general solution of F. We assume a as in (145) so to be taken that the coefficients in Φ are analytic at a. For $|h|$ small, and not zero, (145) is a solution of Φ.

Let q be the highest of the orders of those derivatives of y which appear in Φ. We assume p of §§69, 70 so taken that $p > q$. Now, referring to (145), let

$$\beta = \alpha_0(h) + \alpha_1(h)(x-a) + \cdots + \frac{\alpha_{p+2}(h)}{(p+2)!}(x-a)^{p+2}. \tag{146}$$

We write

$$\beta = \sum_{k=r}^{\infty} \varphi_k(x-a)h^k, \tag{147}$$

with each φ a polynomial in $x - a$ of degree $p + 2$ at most. As α_0 contains h^r, φ_r is not zero for $x = a$.

When β is substituted for y in any form G in Φ, G goes over into a series

$$\gamma = \sum_{k=r}^{\infty} \psi_k(x)h^k \tag{148}$$

with the ψ analytic for $x = a$. We say that any ψ which is not identically zero has a zero at a of order at least $p - q$.

Replacing y in G by $\beta + v$, with v a new unknown and β written as in (146), we get a polynomial G' in v and its derivatives whose coefficients are of the type

$$\sum_{k=0}^{\infty} \delta_k(h)(x-a)^k$$

with each δ analytic for $h = 0$. G' vanishes when v is replaced by the remainder after $p + 3$ terms in (145). When v is thus replaced, a term of G' involving v or a derivative of v yields a series in $x - a$ in which the lowest exponent of $x - a$ is at least $p - q + 3$. Consequently the term in G' free of v and its derivatives, which is the result of substituting β into G, contains no exponent of $x - a$ less than $p - q + 3$. This proves our statement.

72. We shall derive a differential equation of the first order which is satisfied by β in (147) for $|h|$ small. Representing β by y, we write

$$y = \sum_{k=r}^{\infty} \varphi_k(x-a)h^k. \tag{149}$$

Because φ_r is not zero for $x = a$, the r^{th} roots of y are analytic for $x = a$, $h = 0$. We choose some r^{th} root of y and write it

$$y^{1/r} = \sum_{k=1}^{\infty} \xi_i h^k \tag{150}$$

with the ξ analytic at $x = a$ and $\xi_1(a) \neq 0$. The implicit function theorem shows that h is analytic in x and $y^{1/r}$ for $x = a$, $y^{1/r} = 0$. Let then

$$h = \sum_{k=1}^{\infty} \lambda_i y^{k/r} \tag{151}$$

where the λ are functions of x analytic at a. The absence of a term free of y in (151) is explained by the fact that (150) is satisfied by $y^{1/r} = 0$, $h = 0$, for any x close to a.

The derivative of y is given by

$$y_1 = \sum_{k=r}^{\infty} \varphi_k'(x - a)h^k \tag{152}$$

with φ_k' the derivative of φ_k. Replacing h in (152) by its expression in (151), we find a differential equation

$$y_1 = \mu_0 y + \mu_1 y^{1+1/r} + \cdots + \mu_k y^{1+k/r} + \cdots \tag{153}$$

with the μ functions of x analytic at a, which is satisfied by β for $|h|$ small.

73. Differentiating (153) with respect to x and replacing y_1 in the result by its expression in (153), we obtain an expression for y_2 as a series of powers of $y^{1/r}$ which contains no power of y lower than the first power. We obtain similar expressions for $y_3, \cdots, y_q$, where q is as in §71.

Let these expressions be substituted into any form G in Φ. Then G becomes an expression

$$\nu_0 y + \nu_1 y^{1+1/r} + \cdots \tag{154}$$

with each ν analytic at a. We regard the ν as expanded in powers of $x - a$.

We shall prove that the lowest exponent of $x - a$ in any $\nu_k \neq 0$ is at least $p - q$.

Let this be false for ν_0. Then, when we replace y in (154) by its expression in (149), we obtain a series in h in which the coefficient of h^r, as a series in $x - a$, has a least exponent less than $p - q$. This is because φ_r in (149) is not zero for $x = a$. We have thus secured the contradiction that the series in (149), substituted into G, does not yield a ψ_r in (148) with a zero of order at least $p - q$. This accounts for ν_0.

Suppose that the least exponent in ν_1 is less than $p - q$. Then the second term in (154) yields a term in h^{r+1} whose coefficient has a least exponent less than $p - q$. The only other term in (154) which yields a term in h^{r+1} is the first. The least exponent in ν_0 is too great for cancellation to be possible. This accounts for ν_1, and we proceed similarly for the other ν_k.

74. We shall later have occasion to use a sequence of values of p, say p_1, $p_2, \cdots$, increasing to ∞.[108] Let us show that it is possible to use a single a,

[108] Φ, and hence q, will stay fixed.

as above, for all of the p_i. In choosing a for p_1, we are free to take a anywhere in some area. In considering p_2, we limit the domain of x to this area and, in this domain, find an area any point of which will serve for p_2 as well as for p_1. It is now obvious how an a can be obtained which will serve for all p_i. There is a set of such points a dense in the area in which the coefficients in F are meromorphic.

Of course, r in (153) may depend on p_i.

75. For p large, the series in (153) comes close, in a certain sense, to being a y-solution of F in the general solution of F. We know that if the y-solution number of F is ∞, there is a y-solution in the general solution. In the sections which follow, we shall show that if F has a finite y-solution number, F has a y-solution which is approached in a certain manner by the series in (153) as p increases. That y-solution will be in the general solution of F.

Expansions

76. An expression

$$\sum_{k=0}^{\infty} \nu_k y^{k/r} \tag{155}$$

with r a positive integer and the ν functions of x all analytic at some point a, will be called a *y-expansion*. Consider, in (155), a term νy^t with ν not zero. If ν has $s \geqq 0$ zeros at a, the term will be said to be of *degree t* and of *order s*.

All y-expansions occurring in any particular discussion will be understood to involve the same a, but r may be different for different y-expansions.

77. We shall frequently denote y-expansions by a single letter. Let

$$v_1, v_2, \cdots, v_k, \cdots \tag{156}$$

be an infinite sequence of y-expansions. The value of r may change with k and need not even be bounded.

The sequence (156) will be said to *converge to zero* if, given any positive numbers s and t, a positive integer g can be found such that, for $k > g$, either v_k has no term of degree less than t, or else such terms exist and their orders all exceed s.

If there is a y-expansion v such that

$$v - v_1, \cdots, v - v_k, \cdots$$

converges to zero, (156) will be said to *converge to v* or to have v as a *limit*. There can be at most one limit for (156).

78. We consider a sequence (156) which does not converge to zero. There must exist a positive number t[109] for which a positive integer s can be found such

[109] The numbers t which are being described need not be integers.

that there are v_k with k arbitrarily large containing terms of degree less than t which are of order less than s. The greatest lower bound of such numbers t, which, of course, is non-negative, will be called the *characteristic* of the sequence. If (156) converges to zero, its characteristic will be defined as $+\infty$.

79. Let the two sequences of y-expansions

$$v_1, \cdots, v_k, \cdots$$

and

$$w_1, \cdots, w_k, \cdots$$

have respectively the characteristics α and β. Let $\beta > \alpha$.[110] Then the characteristic of

$$v_1 \pm w_1, \cdots, v_k \pm w_k, \cdots$$

is α.

The proof is trivial and will be omitted.

The characteristic of (156) is not greater than that of

$$\frac{\partial v_1}{\partial x}, \cdots, \frac{\partial v_k}{\partial x}, \cdots.$$

Let no v_k in (156) contain a term of degree less than unity. Then the characteristic of (156) exceeds by unity[111] that of

$$\frac{\partial v_1}{\partial y}, \cdots, \frac{\partial v_k}{\partial y}, \cdots.$$

80. Let (156) have a finite characteristic α. If, for every $\epsilon > 0$, there is an $s_\epsilon > 0$ such that *every* v_k with k sufficiently large has a term of degree less than $\alpha + \epsilon$ which is of order less than s_ϵ, we shall call α a *strong* characteristic of the sequence.

If (156) converges to 0, then $+\infty$ will be called a strong characteristic of (156).

We are going to prove that every sequence of y-expansions has a subsequence which possesses a strong characteristic.

Let the least upper bound of the characteristics of all subsequences of (156) be denoted by β. β may be $+\infty$. We shall prove first that (156) has a subsequence of characteristic β. Let γ be any number less than β and s any positive integer. By the nature of β, there are v_k in (156) with k arbitrarily large in which every term of degree less than γ is of order greater than s. Letting γ increase toward β and s increase toward ∞, we secure easily a subsequence of (156) whose characteristic is not less than β and is therefore equal to β.

We shall prove that β is a strong characteristic for the subsequence. Let β

[110] This means that α is finite.

[111] $\infty - 1 = \infty$.

be finite and let there be an $\epsilon > 0$, such that, for every s, there are y-expansions arbitrarily far out in the subsequence for which all terms of degree less than $\beta + \epsilon$ have orders greater than s. Letting s increase, we can form a subsequence of the above subsequence whose characteristic is at least $\beta + \epsilon$. This contradicts the fact that β is the greatest characteristic for all subsequences.

If a sequence has a strong characteristic α, each of its subsequences has α as a strong characteristic.

81. Let the two sequences of y-expansions

$$v_1, \cdots, v_k, \cdots \tag{157}$$

and

$$w_1, \cdots, w_k, \cdots \tag{158}$$

have strong characteristics, equal respectively to α and to β. We shall prove that

$$v_1 w_1, \cdots, v_k w_k, \cdots \tag{159}$$

has a strong characteristic $\alpha + \beta$.

Let one of α, β, say α, be infinite. Then, in v_k with k large, all terms have either a large degree or a large order. The same is therefore true for $v_k w_k$, so that (159) converges to 0.

In what follows, we assume that α and β are finite.

If $\epsilon > 0$, a term of v_k whose degree is less than $\alpha - \epsilon$ will have a high order if k is large. The same holds for terms of w_k of degree less than $\beta - \epsilon$. A term in $v_k w_k$ whose degree is less than $\alpha + \beta - 2\epsilon$ is a sum of products of terms of v_k and w_k, with either the degree of the v_k term less than $\alpha - \epsilon$ or that of the w_k term less than $\beta - \epsilon$. This shows that the characteristic of (159) is not less than $\alpha + \beta$.[112]

To show that the characteristic of (159) does not exceed $\alpha + \beta$, and is strong, is more difficult. Let ϵ be a fixed positive number. Let $s > 0$ be such that for k large, v_k and w_k have terms of orders less than s whose degrees are less respectively than $\alpha + \epsilon$ and $\beta + \epsilon$. We shall produce an integer s' such that $v_k w_k$ with k large has a term of degree less than $\alpha + \beta + 3\epsilon$ whose order is less than s'.

Let δ be a number not less than $\alpha + \beta + 3\epsilon$. Let v_k' and w_k' be respectively the sums of the terms of degree lower than δ in v_k and w_k. Then $v_k' w_k'$ coincides with $v_k w_k$ through terms of degree less than δ.

If v_k', with k large, consists of a single term, that term, by the hypothesis, is of degree less than $\alpha + \epsilon$ and order less than s. As w_k' has a term of degree less than $\beta + \epsilon$ and order less than s, $v_k' w_k'$ will have a term of degree less than $\alpha + \beta + 2\epsilon$ and order less than $2s$. The same result is obtained if w_k' with k large has only one term. In what follows, we assume k to be such that v_k' and w_k' each have more than one term.

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[112] This holds even when the characteristics are not strong.

The equation $v'_k = 0$ is an algebraic equation in $y^{1/r}$ of degree less than $r\delta$ in $y^{1/r}$.[113] The solutions of this equation can be expanded, for x close to a, in ascending rational powers of $x - a$. Given a solution distinct from zero, we shall call the lowest exponent of $x - a$ present effectively in its expansion the *order* of the solution.

We form a Newton polygon which gives the orders of the solutions of $v'_k = 0$. This is done by plotting, for the terms $\nu_i y^{\mu_i}$ in v'_k, the points $(\mu_i r, \sigma_i)$, where σ_i is the order of ν_i; and forming segments of least slope, starting with the leftmost plotted point.

Let the solutions of $v'_k = 0$ distinct from zero be arranged in a sequence so that their orders do not increase. Let the leftmost point[114] on the polygon be (ζ, σ). Let (ζ', σ') be any point on the polygon whose abscissa is an integer. We do not ask that (ζ', σ') be a point plotted for v'_k. Then the sum of the orders of the first $\zeta' - \zeta$ solutions in the above sequence is $\sigma - \sigma'$.[115] This is because a side of the polygon whose slope is $-\rho$ and whose horizontal projection is of length g yields g solutions of order ρ.[116]

Let (ξ, τ) be the left most point on the polygon of w'_k. Then the leftmost point on the polygon of $v'_k w'_k$ is $(\zeta + \xi, \sigma + \tau)$.

Because v'_k has a term of degree (in y) less than $\alpha + \epsilon$ and of order less than s, there must be, on some side of the polygon of v'_k, a point (ζ_1, σ_1) with ζ_1 an integer less than $(\alpha + \epsilon)r$ and $\sigma_1 < s$.[117] The point (ζ_1, σ_1) is not necessarily a point plotted for v'_k.

Similarly, on the polygon of w'_k, there must be a point (ξ_1, τ_1) with ξ_1 an integer less than $(\beta + \epsilon)r$ and $\tau_1 < s$. If the solutions of $w'_k = 0$ distinct from zero are arranged in a sequence so that their orders do not increase, the sum of the first $\xi_1 - \xi$ solutions will be $\tau - \tau_1$.

If now the solutions of $v'_k w'_k = 0$ distinct from zero are arranged as above, the sum of the first

$$(\zeta_1 + \xi_1) - (\zeta + \xi)$$

solutions will be at least $(\sigma + \tau) - (\sigma_1 + \tau_1)$. This means that the polygon of $v'_k w'_k$ has a point,[118] call it h, of abscissa $\zeta_1 + \xi_1$ and of ordinate no more than $\sigma_1 + \tau_1$. We observe that

$$\zeta_1 + \xi_1 < (\alpha + \beta + 2\epsilon)r, \qquad \sigma_1 + \tau_1 < 2s. \tag{160}$$

The point h lies on one or on two sides of the polygon of $v'_k w'_k$. Let l be such a side. Suppose that the slope of l is less than

$$-\frac{2s}{r\epsilon}.$$

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[113] Note that r depends on k.

[114] The plotted point of least abscissa.

[115] If $\zeta' = \zeta$, this means that $\sigma - \sigma' = 0$.

[116] ρ may be negative.

[117] ζ_1 can be taken as the product by r of the degree of the mentioned term of v'_k.

[118] Not necessarily plotted.

Then, if h_1 is the right end of l, the abscissa of h_1 cannot be as great as

$$(\alpha + \beta + 3\epsilon)r.$$

If it were, the ordinate of h_1 would be less than that of h by more than $2s$ and would be negative. Thus, our assumption as to the slope of l implies that $v'_k w'_k$ (also $v_k w_k$) has a term of degree less than $\alpha + \beta + 3\epsilon$ and order less than $2s$.

Suppose now that the slope of l is at least $-2s/r\epsilon$. Let us see how great the ordinate of the left end of l can be. By (160), the intercept of l on the axis of ordinates is less than the quantity s_1 given by

$$s_1 = 2s + \frac{2s}{r\epsilon}(\alpha + \beta + 2\epsilon)r.$$

Then the ordinate of the left end of l is less than s_1. That is, $v'_k w'_k$ contains a term of degree less than $\alpha + \beta + 2\epsilon$ and of order less than s_1.

We notice that s_1, which is actually free of r, depends in no way on k. Let s' be an integer greater than $2s$ and s_1. Then $v_k w_k$ with k large has a term of degree less than $\alpha + \beta + 3\epsilon$ and order less than s'. Thus $\alpha + \beta$ is a strong characteristic for (159).

Completion of Proof

82. We proceed now to carry out the plan sketched in §75. Taking F as in §69, with $y = 0$ in the general solution, we examine the case in which the y-solution number of F is finite. We use Φ as in §71.

According to §73, there exists an infinite sequence of y-expansions

$$(161) \qquad v_1, \cdots, v_k, \cdots$$

with no terms of degree less than unity, such that, when the v_k are substituted into any form G in Φ, we get a sequence of y-expansions converging to zero.

83. Suppose first that (161) converges to zero. We shall prove that $y_1 = 0$ is a y-solution of F in the general solution of F.

It will suffice to prove that if G is any form in Φ, G is annulled by $y_1 = 0$.[119] We shall establish a more general fact, which will be of service later.

Let K be any form in y. Let t be a y-expansion with no terms of degree less than unity and $t_1, t_2, \cdots$, a sequence of such y-expansions converging to 0. Suppose that, when $t + t_k$, $k = 1, 2, \cdots$, is substituted into K, we obtain a sequence which converges to 0. We shall prove that t annuls K.

We know how to substitute

$$(162) \qquad y_1 = t$$

[119] Every form which holds the general solution has a power which is a linear combination of the forms in Φ and their derivatives.

into K. Let us compare the result with the results obtained by making in succession the substitutions

(163) $$y_1 = t + t_k.$$

To distinguish between (162) and (163), let y_1 in (163) be designated by $\bar{y}_1$. We have

$$\bar{y}_1 = y_1 + t_k$$

so that

$$\bar{y}_2 = \frac{\partial}{\partial x}(y_1 + t_k) + (y_1 + t_k)\frac{\partial}{\partial y}(y_1 + t_k)$$

$$= y_2 + \frac{\partial t_k}{\partial x} + (y_1 + t_k)\frac{\partial t_k}{\partial y} + t_k\frac{\partial y_1}{\partial y},$$

or

(164) $$\bar{y}_2 = y_2 + w_k$$

where the w_k approach zero as k increases.

From (164) we find

$$\bar{y}_3 = \frac{\partial}{\partial x}(y_2 + w_k) + (y_1 + t_k)\frac{\partial}{\partial y}(y_2 + w_k)$$

$$= y_3 + z_k$$

where the z_k approach zero as k increases. Similar results are obtained for the higher derivatives.

On this basis, we see that the y-expansions obtained by substituting $t + t_k$ into K converge, as k increases, to the result obtained by substituting t. This shows that t annuls K.

The result stated at the head of this section follows.

84. We assume now that (161) does not converge to zero. Then the characteristic of (161) is at least unity.

If, in the $H(u)$ associated with F, we replace, for each k, u by v_k, u_x by $\partial v_k/\partial x$ and u_y by $\partial v_k/\partial y$, we obtain a sequence of y-expansions converging to zero, namely the sequence produced by F.

The y-solution number of F is n as in Theorem II. In choosing a point a for the formation of (161), we take a in an area in which F has n y-solutions (multiplicities counted) with coefficients analytic throughout the area.

Without loss of generality, we assume that (161), the sequence of the $\partial v_k/\partial x$ and the sequence of the $\partial v_k/\partial y$ all have strong characteristics.[120] Let the

[120] The sequence of $\partial v_k/\partial y$ will have a strong characteristic if (161) does.

characteristic of (161) be $\rho_1 \geqq 1$. Then a term in $y^\mu u^\beta$ in H produces a sequence of characteristic $\mu + \beta\rho_1$ and a term in

$$y^\mu\, u^\beta\, u_x^\gamma\, u_y^\delta$$

gives a characteristic no less than

$$\mu - \delta + \rho_1(\beta + \gamma + \delta).$$

It is easy now to prove that the polygon of H has sides and that $-\rho_1$ is the slope of a side. If, for instance, the polygon had only one point, say (ζ, σ), the related term in $y^\sigma u^\zeta$ would yield, for (161), a characteristic $\sigma + \rho_1\zeta$, which, according to its interpretation as an intercept, would be less than the characteristic obtained from any other term. The proof that $-\rho_1$ is the slope of a side is analogous to the corresponding proof in §47.[121]

Let the sum of those terms of H which are associated with points on the side of slope $-\rho_1$, written with increasing exponents of u, be

$$(165) \qquad b_1 y^{\sigma_1} u^{\zeta_1} + \cdots + b_s y^{\sigma_s} u^{\zeta_s},$$

with the b analytic at a. Then $\sigma_i + \rho_1\zeta_i$ has the same value, say λ, for all i.

Each term in (165) produces, for (161), a sequence of characteristic λ. The other terms in H give characteristics greater than λ. Hence the sum in (165) must give a sequence with a characteristic greater than λ.

Let $g = \zeta_s - \zeta_1$ and let $\psi_1, \cdots, \psi_g$ be the solutions of the algebraic equation

$$b_1 + b_2\psi^{\zeta_2 - \zeta_1} + \cdots + b_s\psi^g = 0.$$

Then the ψ_i are the coefficients of y^{ρ_1} in the y-solutions of F which start with terms in y^{ρ_1}. The ψ_i are thus analytic at a. We may write (165) in the form

$$(166) \qquad b_s y^{\sigma_s} u^{\zeta_1}(u - \psi_1 y^{\rho_1}) \cdots (u - \psi_g y^{\rho_1}).$$

Every subsequence of (161) has ρ_1 as a strong characteristic. (§80.) Replacing (161) by one of its subsequences if necessary, we assume that, for $i = 1, \cdots, g$, when $\psi_i y^{\rho_1}$ is subtracted from the terms of (161), we get a sequence with a strong characteristic.

Because (165) yields a characteristic greater than λ, there must be a factor $u - \psi_i y^{\rho_1}$ in (166) which yields a characteristic greater than ρ_1.[122] Taking such a factor, we represent ψ_i in it by φ_1. Let

$$v_k' = v_k - \varphi_1 y^{\rho_1}, \qquad k = 1, 2, \cdots.$$

Then if $H'(u_1)$ is obtained by putting

$$u = \varphi_1 y^{\rho_1} + u_1$$

[121] H has no b-points, because the y-solution number of F is finite.

[122] $\lambda = \sigma_s + \rho_1\zeta_s = \sigma_s + \rho_1\zeta_1 + \rho_1 g$.

in H, $H'(u_1)$ yields, for the sequence

$$v'_1, \cdots, v'_k, \cdots, \tag{167}$$

which has a strong characteristic greater than ρ_1, a sequence converging to zero.

85. It follows from §83 that if (167) converges to zero, $y_1 = \varphi_1 y^{\rho_1}$ is a y-solution of F which annuls every form in Φ, that is, a y-solution in the general solution of F.

If (167) has a finite characteristic ρ_2, the polygon of H' must have a side of slope $-\rho_2$ and we find a $\varphi_2 y^{\rho_2}$ with φ_2 analytic at a, which, when subtracted from the terms of some subsequence of (167), produces a sequence

$$v''_1, \cdots, v''_k, \cdots \tag{168}$$

with a strong characteristic greater than ρ_2. The expression

$$\varphi_1 y^{\rho_1} + \varphi_2 y^{\rho_2} \tag{169}$$

is a segment of a y-solution of F. If (168) converges to zero, (169) is a y-solution in the general solution of F.

We continue in this manner. The process may terminate at some stage with the isolation of a y-solution of F in the general solution of F, consisting of a finite number of terms. Otherwise, we find that there is a y-solution, call it v, of F, with an infinite number of terms and with all coefficients analytic at a, such that, for any $t > 0$, $s > 0$, we can find a v_k in (161) with k arbitrarily large, such that $v - v_k$ has no terms of degree less than t which are of order less than s. This means that some subsequence of (161) converges to v and that v is a y-solution in the general solution of F.

We have thus completed the proof of the fact that, if F is an algebraically irreducible form of the second order, containing $y = 0$ in its general solution, F has a y-solution which belongs to the general solution of F.

86. The y-solution obtained by the above limiting process may annul the separant of F, in spite of the fact that (145) does not. Let

$$F = yy_2^2 - y_1^2.$$

Then the manifold of y_1 belongs to the general solution, so that $y = 0$ does. The only y-solution is $y_1 = 0$, which annuls the separant.

Summary of Test

87. Let us summarize the method for testing whether $y = 0$, supposed to annul a given algebraically irreducible form F of the second order, belongs to the general solution $\mathfrak{M}$ of F.

If the terms of lowest degree, in F considered as a polynomial in y, y_1, y_2, are free of y_1, y_2, then $y = 0$ is an essential manifold.

If the terms of lowest degree in F involve y_2, $y = 0$ is in $\mathfrak{M}$.

Let the terms of lowest degree involve y_1 but not y_2. Let n be the highest exponent of y_1 in the terms of lowest degree. If there are no B_i in (39) which contain $y = 0$ in their general solutions, $y = 0$ is in $\mathfrak{M}$.[123] If such B_i exist, we determine m of §56. If $m \neq n$, $y = 0$ is in $\mathfrak{M}$. If $m = n$, $y = 0$ is or is not in $\mathfrak{M}$ according as the y-solution number of F is ∞ or n; whether ∞ or n is the y-solution number is decided by the method of §§57–59.

The General Case

88. We return to §20, where it was a question of deciding whether or not the manifold of a given irreducible form of order zero is contained in the general solution $\mathfrak{M}$ of F.

Let D be the form of order zero. Let η be any solution of D.[124] We adjoin η to the underlying field and decompose F into a product $F_1 \cdots F_r$ with the F_i algebraically irreducible in the enlarged field. The F_i are all of order 2. The factorization is accomplished by the methods of abstract algebra.

$\mathfrak{M}$ will contain η when and only when η is approximable in the usual manner by normal solutions of F. A normal solution of some F_i which annuls no other F_i is normal for F. A normal solution of F is normal for some F_i.

It follows that $\mathfrak{M}$ contains η when and only when η is in the general solution $\mathfrak{M}_i$ of some F_i.

To test whether η is in $\mathfrak{M}_i$, we put

$$y = y' + \eta,$$

whereupon F_i goes over into a form G_i in y'. It then becomes a question of determining whether $y' = 0$ is in the general solution of G_i.

This closes the investigation undertaken in §19.

Examples

89. We give first some examples which illustrate the possible relations between m and n in §56.

Example 1. We present a case in which $\mathfrak{Q}$ has no b-points and $n < m$. Let

$$A = y\,(yy_2 + yy_1 - 2y_1^2) - (y_1 - y)^2.$$

Let F be defined by

$$yF = y_1A^2 - \prod_{j=0}^{4}\left(y_1 - y + \frac{y^2}{x + j}\right). \tag{170}$$

Because the second member of (170) vanishes identically in x, y_1, y_2 for $y = 0$, (170) defines F as a differential polynomial. We use the field of all rational functions of x. The algebraic irreducibility of F is seen as follows. F, which is

[123] One might prefer, before examining (39), to see whether $\mathfrak{Q}$ has b-points.
[124] Cf. footnote [40].

of the second degree in y_2, cannot have a factor of the first degree in y_2, for $F = 0$, considered as an algebraic equation, defines y_2 as a function of x, y, y_1 of two branches. A factor of F free of y_2 would have to be a factor of y_1^4, which is a term in F. As F does not vanish identically in all its arguments for $y_1 = 0$, F is algebraically irreducible. F has $y = 0$ as one of its solutions.

The separant of F is $2yy_1 A$. By (170) the common solutions of F and A annul one of

$$B_j = y_1 - y + \frac{y^2}{x + j}, \qquad j = 0, \cdots, 4.$$

For every j,

$$A = y^2 B'_j + C_j B_j \tag{171}$$

where B'_j is the derivative of B_j and C_j is of the first order. Hence A holds every B_j. By (170) yF holds every B_j and as F holds y, F holds every B_j. By (170) a common solution of F and y_1 annuls one of the B_j. All in all, the singular solutions of F constitute the manifolds of the B_j.

Using §5 and (171), we find that the manifolds of the B_j are essential and that the y-solutions

$$y_1 = y - \frac{y^2}{x + j}$$

have multiplicity unity.

Thus $m = 5$.

The terms of lowest degree in F make up $(y_1 - y)^4$. This shows that $\mathfrak{P}$ consists of one side, which joins $(0, 4)$ to $(4, 0)$. Also $\mathfrak{P}$ (and therefore $\mathfrak{Q}$) has no b-points.

We have $n = 4 < m$. Thus, $y = 0$ belongs to the general solution of F.

Example 2. We present a case in which $n = m$ and $\mathfrak{Q}$ has no b-points, but in which the y-solution number of F is ∞. Let

$$A = yy_2 + yy_1 - 2y_1^2.$$

Let

$$F = A^2 + \prod_{j=0}^{2} (y_1 - y + jy^2). \tag{172}$$

We use the field of all constants. The singular solutions are seen to constitute the manifolds of

$$y_1 - y + jy^2, \qquad j = 0, 1, 2$$

which manifolds are essential. The y-solutions

$$y_1 = y - jy^2$$

have multiplicity unity, so that $m = 3$. The terms of lowest degree in F make up $(y_1 - y)^3$ so that $n = 3$ and $\mathfrak{Q}$ has no b-points. Carrying out the substitution

$u = y + u_1$, we find that (2, 2) is a b-point for $\mathfrak{Q}'$, so that $y = 0$ is in the general solution.

Example 3. We consider a case in which $m = n$ and the y-solution number of F is n.

Let $A = y_1^2 - y^3$ and let A_1 be the derivative of A. Let

$$F = A_1^2 - A$$

The singular solutions form the manifold of A, which is irreducible, and essential for F. The y-solutions

$$y_1 = \pm y^{3/2}$$

are seen immediately to have multiplicity unity. Thus $m = 2$. The possibilities for the first terms of y-solutions are found to be $\pm y^{3/2}$. As the coefficients ± 1, which are the roots of $\varphi_1^2 - 1 = 0$, have multiplicity unity, it follows from §66 that the y-solution number of F is $n = 2$. Thus $y = 0$ is not in the general solution of F.

Example 4. We present a case in which $n > m$ and in which the y-solution number is n.

Let $F = y_2^3 - (y_1 + y)y_1$. The singular solutions form the manifold of y_1, which is essential. Thus $m = 1$. There are no b-points on $\mathcal{P}$. We see by §66 that the y-solution number of F is 2. Of course, $y = 0$ is in the general solution of F.

90. We give an example of a y-solution with constant coefficients which diverges for every value of y except $y = 0$. Let

$$F = y_2^2 - y_1 + y.$$

We shall find a y-solution u with constant coefficients.[125] We must have

$$u^2u_y^2 - u + y = 0. \tag{173}$$

Let

$$u = a_1y + a_2y^2 + \cdots + a_ky^k + \cdots .$$

with constant a_k. Then

$$u_y = a_1 + 2a_2y + \cdots + ka_ky^{k-1} + \cdots .$$

Thus

$$uu_y = a_1^2y + \cdots + [(k + 1)a_1a_k + \cdots]\, y^k + \cdots ,$$

where, in the coefficient of y^k with $k > 2$, the terms omitted have positive coefficients. Then

$$u^2u_y^2 = a_1^4y^2 + \cdots + [2ka_1^3a_{k-1} + \cdots]\, y^k + \cdots .$$

[125] It can be seen from §66 that the y-solution number is unity.

Thus $a_1 = 1$, $a_2 = a_1^4$, and

$$a_k \geqq 2ka_1^3\, a_{k-1}$$

for $k > 2$. Thus $a_k \geqq k!$ for $k > 2$. This proves the divergence for $y \neq 0$.

91. We shall show by examples some of the possibilities, for a singular solution which constitutes an essential manifold, as an envelope of solutions. If there are B in (39), a theorem of Hamburger can be used to show that the solutions in the general solutions of the B are usually envelopes of normal solutions of F, with contact of the second order.

Example 1. Let

$$F = (6yy_2 - 5y_1^2)^3 + 729y^4.$$

The only singular solution is $y = 0$, which is an essential manifold. The general solution, found by putting $y = u^3$, is given by

$$y = [(x - a) + b(x - a)^2]^3$$

with a and b constants. Thus every solution in the general solution has contact of the second order with the singular solution.

Example 2. Let

$$F = y_2^3 - 216y.$$

The only singular solution is $y = 0$, an essential manifold. The singular solution has contact of the second order with the solutions of the two one-parameter families $\pm(x - a)^3$. $F = 0$ gives

$$y_2 = 6y^{1/3}.$$

We multiply by $2y_1$ and integrate. Then

$$y_1^2 = 9y^{4/3} + c$$

with c constant. It follows that no solution of F different from the solutions $\pm(x - a)^3$ can vanish at a point together with its first derivative. Thus the singular solution envelops only the above one-parameter families.

Example 3. Let

$$F = y_1^2 y_2 - 8y.$$

Then $y = 0$ is the only singular solution and is an essential manifold. $F = 0$ gives

$$y_1^4 = 16y^2 + c$$

from which we see that the only solutions which vanish together with their first derivatives for some x are $y = \pm(x - a)^2$. Thus $y = 0$ is not an envelope of other solutions with contact of the second order. The fact that $y = 0$ is an envelope with contact of the first order is noteworthy only because F is of the first degree in y_2.

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PROJECTIVE SCALAR DIFFERENTIAL INVARIANTS

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1. The differential invariants of affine and metric spaces have been studied by means of the differential equations satisfied by these invariants. For such invariants the differential equations form a complete system and this fact is used in obtaining the number of functionally independent absolute scalars. When the components of the normal tensors are used as the independent variables in the differential equations for the affine scalars the number of differential equations remains the same for invariants of any order; a similar remark applies to the use of the components $g_{\alpha\beta}$ of the fundamental metric tensor and its extensions in the case of the differential equations satisfied by the metric differential invariants. Such independent variables were used in two papers by T. Y. Thomas and A. D. Michal in the *Annals of Mathematics*, vol. 28 (1927), pp. 196–236 and pp. 631–688, in the second of which there is to be found a bibliography of the more important papers on this subject; for convenience of reference in the following we shall refer to these two papers as I and II respectively. The above choices of the independent variables is of advantage, since in the discussion of earlier writers the number of differential equations increased with the order of the invariants.

In this paper the projective scalar invariants of an affinely connected space are studied by a method which is in general analogous to that used for the affine and metric invariants. It is shown that the differential equations defining these invariants form complete systems, the independent variables being taken as the components of the projective normal tensors; the number of differential equations is the same for invariants of all orders as in the cases previously treated. We have determined the number of projective scalar invariants of any order and have found in this connection interesting results concerning the non-existence of projective invariants of certain particular orders. Whenever possible the treatment has been shortened by reference to the two papers I and II above mentioned.

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2. **Differential equations for projective scalars.** The term projective scalar will be used to denote an absolute scalar $*S$ which is a function of the $\Gamma^i_{ab}(x^1, \cdots, x^n)$[1] and their derivatives, and which remains unaltered in form by the projective change of affine connection

[1] Unless otherwise indicated, Latin indices can take on the values $1, 2, \cdots, n$; and Greek indices the values $0, 1, 2, \cdots, n$.

$$\bar{\Gamma}^i_{ab} = \Gamma^i_{ab} + \delta^i_a \phi_b + \delta^i_b \phi_a ,$$

ϕ_a representing the components of an arbitrary covariant vector.

It has been shown[2] that $*S$ can be expressed as a function of the $\Pi^i_{ab}(x^1, \cdots, x^n)$ and their derivatives, where

$$\Pi^i_{ab} = \Gamma^i_{ab} - \frac{1}{n+1} (\delta^i_a \Gamma^k_{kb} + \delta^i_b \Gamma^k_{ka}) .$$

Also, it has been shown[3] that the latter components Π and their derivatives in $*S$ can be replaced by the components $*A$ of the projective normal tensors. The components $*A^i_{\alpha\beta\gamma_1 \cdots \gamma_p}$ and $*A^0_{\alpha\beta\gamma_1 \cdots \gamma_{p-1}}$ involve derivatives of order $r(\leqq p)$ of the Π's. A scalar $*S$ which involves derivatives of order $\leqq p$ in the Γ's and hence in the Π's will be said to be of order p. It is understood that the derivatives of order p must be present. Then, any $*S$ of order p ($\geqq 1$) can be expressed in terms of components $*A$ of order r ($\leqq p$), so $*S$ (of order p) can be written in the form

$$*S(*A^\sigma_{\alpha\beta\gamma_1}; \cdots ; *A^\sigma_{\alpha\beta\gamma_1 \cdots \gamma_{p-1}}; *A^m_{\alpha\beta\gamma_1 \cdots \gamma_p}) .$$

To obtain the required differential equations for $*S$ we proceed as in I p. 204, and assume $*S(*A)$ retains its form as a function of the $*\bar{A}$ under the infinitesimal transformation

$$x^\alpha = \bar{x}^\alpha + \epsilon \xi^\alpha(\bar{x}^1, \cdots , \bar{x}^n) ,$$

where by definition

$$\epsilon \xi^0(\bar{x}) = \log \begin{vmatrix} \dfrac{\partial x^1}{\partial \bar{x}^1} \cdots \dfrac{\partial x^1}{\partial \bar{x}^n} \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ \dfrac{\partial x^n}{\partial \bar{x}^1} \cdots \dfrac{\partial x^n}{\partial \bar{x}^n} \end{vmatrix} .$$

On expanding $*S(*\bar{A})$ about $\epsilon = 0$, we obtain

$$*S(*\bar{A}) = *S(*A) + \left[\frac{d*S(*\bar{A})}{d\epsilon}\right]_{\epsilon=0} \cdot \epsilon + \cdots .$$

In the above expansion, the coefficient of ϵ is obtained from

$$\left[\frac{d*S(*\bar{A})}{d\epsilon}\right]_{\epsilon=0} = \sum \frac{\partial *S(*A)}{\partial *A} \cdot \left[\frac{d*\bar{A}}{d\epsilon}\right]_{\epsilon=0} ,$$

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[2] J. M. Thomas, *Trans. Amer. Math. Soc.*, vol. 28 (1926), p. 667. Also, J. Douglas, *Annals of Math.*, vol. 29 (1927), p. 160.

[3] T. Y. Thomas, *Annals of Math.*, vol. 28 (1927), p. 549. This paper will be referred to as R.

where the $\sum$ represents the terms obtained by summing on the indices of the components of the various $*A$'s entering in $*S$. The values of $(d^*\bar{A}/d\epsilon)_{\epsilon=0}$ are obtained by differentiating the law of transformation of the $*A$ with respect to ϵ.

Due to the scalar character of $*S$ we must have

$$*S(*\bar{A}) = *S(*A) ,$$

and on setting the coefficient of ϵ equal to zero we find the above invariant condition on $*S$ is obtained if, and only if, the differential equations

$$\begin{aligned} X^t_\sigma(p)\, *S = \sum_{m=1}^{p-1} {}^*\begin{pmatrix} \mu & & t \\ \alpha\beta\gamma_1 & \cdots & \gamma_m\ \sigma \end{pmatrix} \frac{\partial^* S}{\partial^* A^{\mu}_{\alpha\beta\gamma_1\cdots\gamma_m}} \\ + {}^*\begin{pmatrix} i & & t \\ \alpha\beta\gamma_1 & \cdots & \gamma_p\ \sigma \end{pmatrix} \frac{\partial^* S}{\partial^* A^{i}_{\alpha\beta\gamma_1\cdots\gamma_p}} = 0, \end{aligned} \tag{2.1}$$

are satisfied.

These equations are for $*S$ of order $p(\geqq 1)$. The coefficients in these equations are defined by

$$ {}^*\begin{pmatrix} \alpha & & t \\ \beta\gamma\epsilon_1 & \cdots & \epsilon_m\ \sigma \end{pmatrix} = {}^*A^{\alpha}_{\sigma\gamma\epsilon_1\cdots\epsilon_m}\cdot\delta^t_\beta + \cdots {}^*A^{\alpha}_{\beta\gamma\epsilon_1\cdots\sigma}\cdot\delta^t_{\epsilon_m} - {}^*A^{t}_{\beta\gamma\epsilon_1\cdots\epsilon_m}\cdot\delta^{\alpha}_{\sigma}. \tag{2.2}$$

For the affine invariants the corresponding calculations are carried out in I pp. 203–205.

To show that (2.1) form a complete system, we form the commutator

$$[X^t_\sigma, X^s_\tau]*S = X^t_\sigma(X^s_\tau *S) - X^s_\tau(X^t_\sigma *S),$$

and by a calculation similar to that in I p. 207, it is found that

$$[X^t_\sigma, X^s_\tau]*S = \delta^t_\tau X^s_\sigma *S - \delta^s_\sigma\cdot X^t_\tau *S,$$

which shows the completeness.

3. **Independence of the differential equations.** We shall now prove the two theorems.

[3.1]. *If the equations* (2.1) *for* $p \geqq 2$ *are independent, those of order* $p+1$ *are independent.*

[3.2]. *If the equations* (2.1) *for* $p = 2$ *are dependent, those for* $p = 1$ *are dependent.*

The proof of [3.1] follows that of theorem VII of I, and depends on the fact that the number of equations is the same for p and $p + 1$, the value of n being the same in each case. Also those of order $p + 1$ are obtained from those of order p by adding terms which do not affect the independence of the equations of order p.

To prove [3.2], consider (2.1) for $p = 2$ and $p = 1$, $(n > 2)$, i.e.,

$$*\begin{pmatrix}\mu & t\\ \alpha\beta\gamma & \sigma\end{pmatrix}\frac{\partial^* S}{\partial^* A^{\mu}_{\alpha\beta\gamma}} + *\begin{pmatrix}i & t\\ \alpha\beta\gamma\delta & \sigma\end{pmatrix}\frac{\partial^* S}{\partial^* A^{i}_{\alpha\beta\gamma\delta}} = 0, \tag{3.1}$$

$$*\begin{pmatrix}i & t\\ abc & s\end{pmatrix}\frac{\partial^* S}{\partial^* A^{i}_{abc}} = 0. \tag{3.2}$$

By the hypothesis that (3.1) are dependent, the equations

$$\lambda^{\sigma}_{t}\, *\begin{pmatrix}\mu & t\\ \alpha\beta\gamma & \sigma\end{pmatrix} = 0, \tag{3.3a}$$

$$\lambda^{\sigma}_{t}\, *\begin{pmatrix}i & t\\ \alpha\beta\gamma\delta & \sigma\end{pmatrix} = 0, \tag{3.3b}$$

have solutions λ^{σ}_{t} not all identically zero. For $\mu \neq 0$, the equations (3.3a) reduce to

$$\lambda^{s}_{t}\, *\begin{pmatrix}i & t\\ abc & s\end{pmatrix} = 0. \tag{3.3c}$$

In obtaining (3.3c) we use the fact[4] that

$$*A^{i}_{\alpha\beta\gamma} = 0, \qquad (\alpha,\ \beta, \text{ or } \gamma = 0).$$

Now if (3.3c) are satisfied with λ^{s}_{t} not all identically zero, then (3.2) are dependent. So we assume then that $\lambda^{s}_{t} = 0$. Then (3.3a) for $\mu = 0$ becomes

$$\lambda^{0}_{t}\, *\begin{pmatrix}0 & t\\ \alpha\beta\gamma & 0\end{pmatrix} = 0,$$

which reduces to

$$*A^{t}_{abc}\cdot\lambda^{0}_{t} = 0. \tag{3.4}$$

Now for the general manifold, i.e., one for which the $*A^{i}_{abc}$ are not connected by any relations except their complete set of identities, there will exist for $n > 2$, a determinant A of order n formed from the matrix $\| *A^{t}_{abc} \|$, such that

$$A \not\equiv 0, \tag{3.5}$$

and hence $\lambda^{0}_{t} = 0$, which contradicts the hypothesis that (3.1) were dependent. Hence, under (3.5) the equations (3.3c) possess a solution λ^{s}_{t} with these quantities not all zero. That is, assuming (3.5), there exists a set D of determinants of the matrix of the coefficients of the unknowns λ in (3.3c) which vanish. As this set of determinants and A are polynomials in the $*A$, and the set D vanishes whenever R does not, it follows by a theorem in algebra[5] that the set D vanishes whether A does or not. This proves the theorem. The case $n = 2$ will be treated later.

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[4] R, p. 560.

[5] Bôcher, *Introduction to Higher Algebra*, p. 8.

The following theorem will now be proved.

[3.3]. *If the equations* (2.1) *for* $p = 1$ *are independent for the general affine space of* n *dimensions, they are independent for the general affine space of* $n + 1$ *dimensions.*

From the matrix of the coefficients of the equations (2.1) for $p = 1$ with reference to $n + 1$ dimensions, namely

$$\text{(3.6)} \qquad {}^*\begin{pmatrix} i & t \\ jkl & s \end{pmatrix} \frac{\partial^* S}{\partial^* A^{\,i}_{jkl}} = 0, \qquad (i, j, k, l, t, s = 1, 2, \cdots, n, n+1),$$

we can select a determinant M of order $(n + 1)^2$ which is not identically zero as shown by the following considerations.

Choose

$$\text{(3.7)} \qquad \begin{aligned} {}^*A^{n+1}_{i\,j\,k} &= 0, \\ {}^*A^{\,i}_{jk\;n+1} &= 0, \\ {}^*A^{n+1}_{n+1\;n+1\;i} &= 0, \\ {}^*A^{\,i}_{n+1\;jk} &= 0, \\ {}^*A^{\,i}_{i\;n+1\;n+1} &= 0, \ (i \text{ not summed}). \end{aligned}$$

These imply on account of the identities satisfied by the components $^*A$ in (3.7)

$$^*A^{n+1}_{n+1\;i\;n+1} = 0, \qquad ^*A^{\,i}_{n+1\;n+1\;i} = 0, \qquad (i \text{ not summed}).$$

(In these latter equations and in (3.7), $i, j, k, = 1, \cdots, n$.)

The other $^*A$ are to be arbitrary subject to their identities. From the above choices we obtain

$$\text{(3.8)} \qquad \begin{aligned} &{}^*\begin{pmatrix} i & t \\ jkl & n+1 \end{pmatrix} = 0, \qquad {}^*\begin{pmatrix} i & & t \\ jk & n+1 & s \end{pmatrix} = 0, \\ &{}^*\begin{pmatrix} i & & & n+1 \\ n+1 & n+1 & j & s \end{pmatrix} = 0, \qquad {}^*\begin{pmatrix} i & n+1 \\ jkl & s \end{pmatrix} = 0, \\ &{}^*\begin{pmatrix} i & & n+1 \\ jk & n+1 & n+1 \end{pmatrix} = 0, \qquad {}^*\begin{pmatrix} i & n+1 \\ jkl & n+1 \end{pmatrix} = 0, \\ &{}^*\begin{pmatrix} i & & & t \\ n+1 & n+1 & j & n+1 \end{pmatrix} = 0, \qquad (i, j, k, l, t, s = 1, \cdots, n). \end{aligned}$$

The determinant M is chosen as shown in Table I.

In this table the position of the element ${}^*\begin{pmatrix} i & t \\ jkl & s \end{pmatrix}$ is indicated by the symbol ${}^*\begin{pmatrix} i \\ jkl \end{pmatrix}$ at the left which designates the row, and the symbol $\begin{pmatrix} t \\ s \end{pmatrix}$ at the top which designates the column. Similar remarks apply to the remaining elements

TABLE I

	$\binom{t}{s}$	$\binom{t}{n+1}$	$\binom{n+1}{s}$	$\binom{n+1}{n+1}$
${}^*\binom{i}{jkl}$	Δ_1	0	0	0
${}^*\binom{i}{jk\ n+1}$	0	Δ_2		0
${}^*\binom{i}{n+1\ n+1\ j}$		0	0	Δ_3

of M. The squares marked Δ_1, Δ_2, Δ_3 represent determinants of orders n^2, $2n$, and 1, respectively. The letters i, j, k, l, t, s in Table I correspond to values $1, 2, \cdots, n$ in (3.6). The zeros in Table I follow from the relations (3.8).

We choose Δ_1 to be a non-vanishing determinant of order n^2 for n dimensions, the existence of Δ_1 being implied by the hypothesis of the theorem.

For Δ_2 we pick ${}^*\binom{i}{jk\ n+1}$ corresponding to the indices

$$
{}^*\binom{1}{11\ n+1}, {}^*\binom{2}{22\ n+1}, \cdots, {}^*\binom{n}{nn\ n+1}, {}^*\binom{1}{22\ n+1}, \\
{}^*\binom{2}{33\ n+1}, \cdots, {}^*\binom{n}{11\ n+1}.
$$

With the aid of (3.7), Δ_2 now has the form shown in Table II.

To get a non-vanishing Δ_3 we take $i = 1, j = 2$ for the last row of Table I, giving

$$
{}^*\binom{1 \qquad\qquad\quad n+1}{n+1\ n+1\ 2\ n+1} = 2\ {}^*A^{1}_{n+1\ n+1\ 2} = \Delta_3 .
$$

The determinant M then has the value $\Delta_1 \cdot \Delta_2 \cdot \Delta_3 \not\equiv 0$ which shows the independence of the equations (3.6) and thus completes the proof of [3.3].

4. **Number of independent projective normal tensors.** It has been shown[6] that the identities existing between the components of the projective normal tensors ${}^*A$ can be found by the use of a set of non-tensor invariants Q. The Q's are the coefficients in the power series expansion of the projective connec-

[6] R, p. 560. This section follows that paper.

TABLE II

	$\binom{1}{n+1}$	$\binom{2}{n+1}$	$\cdots$	$\binom{n}{n+1}$	$\binom{n+1}{1}$	$\binom{n+1}{2}$	$\cdots$	$\binom{n+1}{n}$
$^*\binom{1}{11\ n+1}$	0	0	$\cdots$	0	$-{}^*A^{n+1}_{11\ n+1}$	$^*A^{1}_{112}$	$\cdots$	$^*A^{1}_{11n}$
$^*\binom{2}{22\ n+1}$	0	0	$\cdots$	0	$^*A^{2}_{221}$	$-{}^*A^{n+1}_{22\ n+1}$	$\cdots$	$^*A^{2}_{22n}$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$^*\binom{n}{nn\ n+1}$	0	0	$\cdots$	0	$^*A^{n}_{nn1}$	$^*A^{n}_{nn2}$	$\cdots$	$-{}^*A^{n+1}_{nn\ n+1}$
$^*\binom{1}{22\ n+1}$	0	$2{}^*A^{1}_{2\ n+1\ n+1}$	$0 \cdots$	0				
$^*\binom{2}{33\ n+1}$	0	0	$2{}^*A^{2}_{3\ n+1\ n+1}$ $0 \cdots$	0				
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$				
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$				
$^*\binom{n-1}{nn\ n+1}$	0	0	$\cdots$	$2{}^*A^{n-1}_{nn\ n+1}$				
$^*\binom{n}{11\ n+1}$	$2{}^*A^{n}_{1\ n+1\ n+1}$	0	$\cdots$	0				

The elements in the lower right-hand square are not needed. Hence, to within sign,

$$\Delta_2 = 2^n \cdot {}^*A^{1}_{2\ n+1\ n+1} \cdots {}^*A^{n}_{1\ n+1\ n+1} \cdot \begin{vmatrix} -{}^*A^{n+1}_{11\ n+1} & {}^*A^{1}_{112} & \cdots & {}^*A^{1}_{11n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ {}^*A^{n}_{nn1} & {}^*A^{n}_{nn2} & \cdots & -{}^*A^{n+1}_{nn\ n+1} \end{vmatrix},$$

which does not vanish identically.

tion Π^i_{ab} in projective normal coördinates. From this fact it is shown that the Q's satisfy the complete set of identities

$$Q^i_{abcd\cdots s} = Q^i_{bacd\cdots s}\,, \tag{4.1a}$$

$$Q^i_{abcd\cdots s} = Q^i_{abef\cdots t}\,, \tag{4.1b}$$

$$S(Q^i_{abcd\cdots s}) = 0\,, \tag{4.2}$$

$$Q^i_{ibcd\cdots s} = 0\,. \tag{4.3}$$

In (4.1b), $(ef\cdots t)$ represents any permutation of $(cd\cdots s)$, and in (4.2), S denotes the sum of all terms obtainable by permutation of subscripts from the term in parentheses, only such new terms being added which are not identical on account of (4.1).

The first Q of the set, Q^i_{abc}, is a tensor.

If $P^i_{ab}(y^1, \cdots, y^n)$ represent the components of the projective connection in normal coördinates, then

$$Q^i_{abc_1\cdots c_p}{}^{(q)} = \frac{\partial^p P^i_{ab}(0)}{\partial y^{c_1}\cdots\partial y^{c_p}}\,, \qquad P^i_{ab}(0) = 0\,. \tag{4.4}$$

Also, let $*C^\alpha_{\beta\gamma}(y^1, \cdots, y^n)$ and $*V(y^1, \cdots, y^n)$ denote the components $*\Gamma^\alpha_{\beta\gamma}$ and $*A$ respectively in normal coördinates. The expressions for the $*A$ in terms of the Q's are now found by evaluating at the origin of the normal coördinates the relations which define the $*V$ in terms of $*C^\alpha_{\beta\gamma}$ and the derivatives of $*C^\alpha_\beta$ using

$$*A(q) = *V(0)\,. \tag{4.5}$$

From the identities (4.1), (4.2), (4.3), it follows that the number of functionally independent Q components of order $p(\geqq 1)$, ($p+2$ subscripts), for n dimensions is

$$N(n, p) = nK(n, 2)K(n, p) - nK(n, p+2) - nK(n, p)\,, \tag{4.6}$$

where

$$K(n, p) = \frac{n(n+1)\cdots(n+p-1)}{p\,!}\,.$$

The number $N(n, p)$ of (4.6) gives the number of functionally independent $*A$ of order $p(\geqq 1)$. It is for this reason that the use of the Q's is so convenient.

5. **The case** $n = 3$, $p = 1$. It will be shown that for $n = 3$ and $p = 1$ the equations (2.1) are independent, and then from [3.2], [3.1], [3.3] it will follow that (2.1) are independent for $n \geqq 3$ and $p \geqq 1$.

The relation between Q^i_{abc} and $*A^{\,i}_{abc}$ is found from

$$(5.1)\qquad *V^i_{abc} = \frac{\partial *C^i_{ab}}{\partial y^c} - *C^i_{abc} - *C^i_{\sigma b}*C^i_{ac} - *C^i_{a\sigma}*C^\sigma_{bc},$$

where

$$(5.2)\qquad \begin{aligned} *C^i_{abc} = \frac{1}{3}\left(\frac{\partial *C^i_{ab}}{\partial y^c} + \frac{\partial *C^i_{bc}}{\partial y^a} + \frac{\partial *C^i_{ca}}{\partial y^b}\right) \\ - \frac{2}{3}(*C^i_{\sigma a}*C^\sigma_{bc} + *C^i_{\sigma b}*C^\sigma_{ca} + *C^i_{\sigma c}*C^\sigma_{ab}), \end{aligned}$$

$$(5.3)\qquad Q^i_{abc} = *A^{\,i}_{abc}.$$

From (4.6) we get

$$(5.4)\qquad N(3, 1) = 15$$

and from the identities

$$Q^i_{abc} = Q^i_{bac}; \qquad Q^i_{abc} + Q^i_{bca} + Q^i_{cab} = 0; \qquad Q^i_{ibc} = 0,$$

ti is found that we can take the independent components $*A^{\,i}_{abc}$ to be

$$(5.5)\qquad \begin{gathered} *A^{\,i}_{121} = a^i, \qquad *A^{\,i}_{212} = c^i, \qquad *A^{\,i}_{313} = e^i, \\ *A^{\,1}_{131} = b^1, \qquad *A^{\,2}_{131} = b^2, \qquad *A^{\,1}_{232} = d^1, \qquad *A^{\,2}_{232} = d^2, \\ *A^{\,1}_{323} = f^1, \qquad *A^{\,3}_{323} = f^3. \end{gathered}$$

The remaining distinct components different from zero are

$$(5.6)\qquad \begin{gathered} *A^{\,i}_{112} = -2a^i, \quad *A^{\,i}_{221} = -2c^i, \quad *A^{\,i}_{331} = -2e^i, \quad *A^{\,3}_{113} = 2a^2, \\ *A^{\,3}_{223} = 2c^1, \quad *A^{\,2}_{332} = 2e^1, \quad *A^{\,1}_{123} = 2d^2 - f^3, \quad *A^{\,2}_{123} = 2b^1 - e^3, \\ *A^{\,3}_{123} = -c^2 - a^1, \quad *A^{\,3}_{131} = -a^2, \quad *A^{\,3}_{232} = -c^1, \quad *A^{\,2}_{323} = -e^1, \\ *A^{\,1}_{231} = -(d^2 + f^3), \quad *A^{\,2}_{231} = 2e^3 - b^1, \quad *A^{\,3}_{231} = 2c^2 - a^1, \\ *A^{\,1}_{312} = 2f^3 - d^2, \quad *A^{\,2}_{312} = -(b^1 + e^3), \quad *A^{\,3}_{312} = 2a^1 - c^2, \\ *A^{\,1}_{113} = -2b^1, \quad *A^{\,2}_{223} = -2d^2, \quad *A^{\,3}_{331} = -2e^3, \quad *A^{\,3}_{332} = -2f^3, \\ *A^{\,1}_{223} = -2d^1, \quad *A^{\,1}_{332} = -2f^1, \quad *A^{\,3}_{332} = -2f^3, \quad *A^{\,2}_{113} = -2b^2. \end{gathered}$$

In the differential equations

$$(5.7)\qquad *\begin{pmatrix} i & t \\ abc & s \end{pmatrix} \frac{\partial *S}{\partial *A^{\,i}_{abc}} = 0,$$

we express the coefficients in terms of the 15 independent components and consider $*S$ to be also thus expressed.

These equations then are represented by Table III.

TABLE III

$\binom{l}{s} =$	$\binom{1}{1}$	$\binom{2}{1}$	$\binom{3}{1}$	$\binom{1}{2}$	$\binom{2}{2}$	$\binom{3}{2}$	$\binom{1}{3}$	$\binom{2}{3}$	$\binom{3}{3}$
$\partial/\partial a^1$	a^1	$-a^2$	$-a^3$	$-c^1$	a^1	0	$d^2 - 2f^3$	b^1	0
a^2	$2a^2$	0	0	$-(a^1 + c^2)$	0	$-a^2$	$b^1 + e^3$	b^2	0
a^3	$2a^3$	0	0	$-c^3$	a^3	0	$c^2 - 3a^1$	$-2a^2$	$-a^3$
b^1	b^1	$-b^2$	a^2	f^3-2d^2	0	a^1	$-e^1$	0	b^1
b^2	$2b^2$	0	0	$e^3 - 3b^1$	$-b^2$	$2a^2$	$-e^2$	0	b^2
c^1	0	$-(a^1 + c^2)$	$-c^3$	0	$2c^1$	0	d^1	$d^2 + f^3$	0
c^2	c^2	$-a^2$	0	$-c^1$	c^2	$-c^3$	d^2	$b^1 - 2e^3$	0
c^3	c^3	$-a^3$	0	0	$2c^3$	0	$-2c^1$	$a^1 - 3c^2$	$-c^3$
d^1	$-d^1$	$f^3 - 3d^2$	$2c^1$	0	$2d^1$	0	0	$-f^1$	d^1
d^2	0	$e^3 - 2b^1$	c^2	$-d^1$	d^2	c^1	0	e^1	d^2
e^1	0	$-e^2$	$-(b^1 + e^3)$	f^1	0	$d^2 + f^3$	0	0	$2e^1$
e^2	e^2	0	$-b^2$	$-2e^1$	$-e^2$	$b^1 - 3e^3$	0	0	$2e^2$
e^3	e^3	0	a^2	f^3	0	$a^1 - 2c^2$	$-e^1$	$-e^2$	e^3
f^1	$-f^1$	$2e^1$	$d^2 - 3f^3$	0	f^1	$-d^1$	0	0	$2f^1$
f^3	0	e^3	$c^2 - 2a^1$	0	f^3	c^1	$-f^1$	e^1	f^3

To show the existence of a non-vanishing determinant of order 9, we select the last 9 rows and put each variable equal to one. This gives

$$\begin{vmatrix} 0 & 1 & -1 & 0 & 1 & 1 & -1 & 1 & 1 \\ -1 & 2 & -2 & 0 & 1 & -1 & 0 & 0 & 2 \\ 1 & 0 & 1 & 1 & 0 & -1 & -1 & -1 & 1 \\ 1 & 0 & -1 & -2 & -1 & -2 & 0 & 0 & 2 \\ 0 & -1 & -2 & 1 & 0 & 2 & 0 & 0 & 2 \\ 0 & -1 & 1 & -1 & 1 & 1 & 0 & 1 & 1 \\ -1 & -2 & 2 & 0 & 2 & 0 & 0 & -1 & 1 \\ 1 & -1 & 0 & 0 & 2 & 0 & -2 & -2 & -1 \\ 1 & -1 & 0 & -1 & 1 & -1 & 1 & -1 & 0 \end{vmatrix} = 448.$$

6. **The case** $n = 2$. From (4.6) we find

$$N(2, p) = 2(p - 1). \tag{6.1}$$

Hence we have

[6.1] *For $n = 2$ there are no projective scalars of the first order.*

Since there are now 6 differential equations in (2.1), and $N(2, 2) + N(2, 3) = 6$,

we must consider these equations for at least $p = 2$ and $p = 3$. We first take $p = 2$.

It is found that for a general n

$$
\begin{aligned}
&{}^*A^{i}_{abc} = Q^{i}_{abc}, \\
(6.2)\qquad &{}^*A^{0}_{abc} = \frac{n+1}{n-1} Q^{i}_{abci}, \\
&{}^*A^{\mu}_{\alpha\beta\gamma} = 0, \quad \text{(otherwise)}
\end{aligned}
$$

$$
(6.3)\qquad {}^*A^{i}_{abcd} = Q^{i}_{abcd} - \frac{1}{n+1}(\delta^i_c {}^*A^{0}_{abd} + \delta^i_d {}^*A^{0}_{abc}).
$$

To determine the remaining ${}^*A^{i}_{\alpha\beta\gamma\delta}$ in terms of the Q's, we must use the equations

$$
\begin{aligned}
{}^*V^{\mu}_{\alpha\beta\gamma\delta} = {}& \frac{\partial^2 {}^*C^{\mu}_{\alpha\beta}}{\partial y^\gamma \partial y^\delta} - {}^*C^{\mu}_{\alpha\beta\gamma\delta} - \frac{\partial {}^*C^{\mu}_{\alpha\sigma}}{\partial y^\gamma} {}^*C^{\sigma}_{\beta\delta} - \frac{\partial {}^*C^{\mu}_{\sigma\beta}}{\partial y^\gamma} \cdot {}^*C^{\sigma}_{\alpha\delta} - \frac{\partial {}^*C^{\mu}_{\alpha\sigma}}{\partial y^\delta} {}^*C^{\sigma}_{\beta\gamma} \\
(6.4)\qquad & - \frac{\partial {}^*C^{\mu}_{\sigma\beta}}{\partial y^\delta} \cdot {}^*C^{\sigma}_{\alpha\gamma} - \frac{\partial {}^*C^{\mu}_{\alpha\beta}}{\partial y^\sigma} \cdot {}^*C^{\sigma}_{\gamma\delta} - {}^*C^{\mu}_{\alpha\sigma} \cdot {}^*C^{\sigma}_{\beta\gamma\delta} - {}^*C^{\mu}_{\sigma\beta} {}^*C^{\sigma}_{\alpha\gamma\delta} \\
& + {}^*V^{\sigma}_{\alpha\beta\gamma} \cdot {}^*C^{\mu}_{\sigma\delta} + {}^*V^{\sigma}_{\alpha\beta\gamma} \cdot {}^*C^{\mu}_{\sigma\gamma} + {}^*C^{\mu}_{\sigma\epsilon}({}^*C^{\sigma}_{\alpha\gamma} {}^*C^{\epsilon}_{\beta\delta} + {}^*C^{\sigma}_{\alpha\delta} {}^*C^{\epsilon}_{\beta\gamma}),
\end{aligned}
$$

where

$$
\begin{aligned}
{}^*C^{\mu}_{\alpha\beta\gamma_1\cdots\gamma_p} = {}& \frac{1}{p+2} \cdot P\left(\frac{\partial {}^*C^{\mu}_{\alpha\beta\gamma_1\cdots\gamma_{p-1}}}{\partial y^{\gamma_p}} - {}^*C^{\mu}_{\tau\beta\gamma_1\cdots\gamma_{p-1}} {}^*C^{\tau}_{\alpha\sigma} - \cdots \right. \\
(6.5)\qquad & \left. - {}^*C^{\mu}_{\alpha\beta\gamma_1\cdots\tau} {}^*C^{\tau}_{\gamma_{p-1}\gamma_p}\right),
\end{aligned}
$$

and P denotes the sum of all the terms obtainable from the one in the parentheses by cyclic permutations of the indices $(\alpha\beta\gamma_1 \cdots \gamma_p)$; so ${}^*C^{\mu}_{\alpha\beta\gamma\cdots\delta}$ is symmetric in its lower indices.

By the use of (6.5), (4.2), and (4.4), it is found that

$$
\begin{aligned}
(6.6)\qquad & {}^*C^{i}_{0bcd}(0) = 0, \qquad {}^*C^{i}_{00cd}(0) = 0, \qquad {}^*C^{i}_{000d}(0) = 0, \\
& {}^*C^{i}_{0000}(0) = 0, \qquad {}^*C^{i}_{abcd}(0) = 0,
\end{aligned}
$$

and thus by evaluating (6.4) at the origin of the $(y^0, y^1, \cdots, y^n)$ coördinate system, that

$$
\begin{aligned}
& {}^*A^{i}_{0bcd} = \frac{-1}{n+1} Q^{i}_{cdb}, && {}^*A^{i}_{a000} = \frac{-6}{(n+1)^3} \cdot \delta^i_a, \\
(6.7)\qquad & {}^*A^{i}_{00cd} = 0, && {}^*A^{i}_{ab00} = 0, \\
& {}^*A^{i}_{000d} = \frac{-6}{(n+1)^3} \cdot \delta^i_d, && {}^*A^{i}_{abc0} = \frac{2}{n+1} Q^{i}_{abc}, \\
& {}^*A^{i}_{0000} = 0, && {}^*A^{i}_{0b0d} = 0.
\end{aligned}
$$

The remaining $*A^{i}_{\alpha\beta\gamma\delta}$ with at least one lower index zero are found from those given above with the aid of the fact that $*A^{\mu}_{\alpha\beta\gamma\delta}$ is symmetric in α, β and γ, δ.

By the use of (4.1), (4.2), and (4.3) for $n = 2$, it is found that we can take as independent

$$Q^1_{1112} = a, \qquad Q^2_{2221} = b, \tag{6.8}$$

and the remaining distinct components are

$$\begin{aligned}
&Q^1_{2111} = -a, && Q^1_{1221} = -b, && Q^2_{2111} = 0, && Q^i_{1111} = 0,\\
&Q^2_{2112} = -a, && Q^2_{1222} = -b, && Q^1_{1222} = 0, && Q^i_{2222} = 0,\\
&Q^2_{2211} = a, && Q^1_{1122} = b, && Q^2_{1112} = 0, &&\\
&Q^2_{1122} = 3a, && Q^1_{2211} = 3b, && Q^1_{2221} = 0. &&
\end{aligned} \tag{6.9}$$

Also

$$Q^i_{abc} = 0, \qquad (n = 2). \tag{6.10}$$

So from (6.2) we get

$$\begin{aligned}
&*A^{i}_{abc} = 0, && (n = 2), && &&\\
&*A^{0}_{111} = 0, && *A^{0}_{222} = 0, && &&\\
&*A^{0}_{121} = -6a, && *A^{0}_{212} = -6b, && *A^{0}_{112} = 12a, && *A^{0}_{221} = 12b;
\end{aligned} \tag{6.11}$$

and then from (6.3)

$$\begin{aligned}
&*A^{1}_{1112} = -3a, && *A^{2}_{1112} = 0, && *A^{1}_{1211} = 3a, && *A^{2}_{1211} = 0,\\
&*A^{2}_{1221} = a, && *A^{2}_{2211} = a, && *A^{2}_{1122} = -5a, && *A^{1}_{1122} = b,\\
&*A^{1}_{1221} = b, && *A^{1}_{2221} = 0, && *A^{2}_{2221} = -3b, && *A^{1}_{2122} = 0,\\
&*A^{2}_{2122} = 3b, && *A^{1}_{2211} = -5b. && &&
\end{aligned} \tag{6.12}$$

On putting $-6a = \alpha$, $-6b = \beta$, the differential equations (2.1) for $p = 2$ take the form

$$\begin{aligned}
X^1_0&: \quad 0\cdot\frac{\partial}{\partial\alpha} + 0\cdot\frac{\partial}{\partial\beta} = 0,\\
X^1_1&: \quad 2\alpha\frac{\partial}{\partial\alpha} + \beta\frac{\partial}{\partial\beta} = 0,\\
X^1_2&: \quad -\beta\frac{\partial}{\partial\alpha} + 0\cdot\frac{\partial}{\partial\beta} = 0,\\
X^2_0&: \quad 0\cdot\frac{\partial}{\partial\alpha} + 0\cdot\frac{\partial}{\partial\beta} = 0,\\
X^2_1&: \quad 0\cdot\frac{\partial}{\partial\alpha} - \alpha\frac{\partial}{\partial\beta} = 0,\\
X^2_2&: \quad \alpha\cdot\frac{\partial}{\partial\alpha} + 2\beta\frac{\partial}{\partial\beta} = 0.
\end{aligned} \tag{6.13}$$

From X^1_2 and X^2_1, it is seen that

$$\frac{\partial{}^*S}{\partial\alpha} = 0, \qquad \frac{\partial{}^*S}{\partial\beta} = 0,$$

from which we have

[6.2]. *For $n = 2$ there are no projective scalars of the second order.*

The case $n = 2$, $p = 3$ will now be discussed.

We must first obtain some relations holding for a general n. The expression for the components ${}^*V^{\sigma}_{\alpha\beta\gamma\delta\epsilon}$ in the system of coördinates $(y^0, y^1, \cdots, y^n)$ is

$$\begin{aligned} {}^*V^{\sigma}_{\alpha\beta\gamma\delta\epsilon} = {} & \frac{\partial^3\, {}^*C^{\sigma}_{\alpha\beta}}{\partial y^{\gamma}\,\partial y^{\delta}\,\partial y^{\epsilon}} - {}^*C^{\sigma}_{\alpha\beta\gamma\delta\epsilon} - {}^*C^{\tau}_{\beta\gamma\delta\epsilon}\,{}^*C^{\sigma}_{\alpha\tau} - {}^*C^{\tau}_{\alpha\gamma\delta\epsilon}\,{}^*C^{\sigma}_{\beta\tau} - {}^*C^{\tau}_{\gamma\delta\epsilon}\,\frac{\partial{}^*C^{\sigma}_{\alpha\beta}}{\partial y^{\tau}} \\ & + \underset{\gamma,\delta,\epsilon}{S}\Bigg(- \frac{\partial^2\,{}^*C^{\sigma}_{\alpha\beta}}{\partial y^{\gamma}\,\partial y^{\tau}}\,{}^*C^{\tau}_{\delta\epsilon} - \frac{\partial^2\,{}^*C^{\sigma}_{\alpha\tau}}{\partial y^{\gamma}\,\partial y^{\delta}}\,{}^*C^{\tau}_{\beta\epsilon} - \frac{\partial^2\,{}^*C^{\sigma}_{\beta\tau}}{\partial y^{\gamma}\,\partial y^{\delta}}\cdot{}^*C^{\tau}_{\alpha\epsilon} + \frac{\partial{}^*C^{\sigma}_{\lambda\tau}}{\partial y^{\sigma}}\cdot{}^*C^{\lambda}_{\alpha\delta}\,{}^*C^{\tau}_{\beta\epsilon} \\ & + \frac{\partial{}^*C^{\sigma}_{\alpha\lambda}}{\partial y^{\tau}}\,{}^*C^{\lambda}_{\beta\epsilon}\,{}^*C^{\tau}_{\gamma\delta} + \frac{\partial{}^*C^{\sigma}_{\beta\lambda}}{\partial y^{\tau}}\cdot{}^*C^{\lambda}_{\alpha\epsilon}\,{}^*C^{\tau}_{\gamma\delta} - \frac{\partial{}^*C^{\sigma}_{\alpha\tau}}{\partial y^{\gamma}}\cdot{}^*C^{\tau}_{\beta\delta\epsilon} - \frac{\partial{}^*C^{\sigma}_{\beta\tau}}{\partial y^{\gamma}}\cdot{}^*C^{\tau}_{\alpha\delta\epsilon} \\ & + {}^*C^{\sigma}_{\lambda\tau}\,[{}^*C^{\tau}_{\beta\epsilon}\,{}^*C^{\lambda}_{\alpha\gamma\delta} + {}^*C^{\tau}_{\alpha\epsilon}\,{}^*C^{\lambda}_{\beta\gamma\delta}] + {}^*V^{\tau}_{\alpha\beta\gamma}\,{}^*C^{\sigma}_{\tau\gamma\epsilon} + {}^*V^{\tau}_{\alpha\beta\gamma\delta}\cdot{}^*C^{\sigma}_{\tau\epsilon}\Bigg), \end{aligned} \tag{6.14}$$

where $\underset{\gamma,\delta,\epsilon}{S}$ denotes the sum of all terms obtainable by permutations of the subscripts γ, δ, ϵ which do not give equivalent terms.

From (4.4), (5.2), and (6.5), we find

$$\begin{aligned} {}^*C^{\alpha}_{\beta 0 0}(0) &= \frac{-\,2\delta^{\alpha}_{\beta}}{(n+1)^2}, \\ {}^*C^{\alpha}_{\beta\gamma\delta}(0) &= 0, \\ \frac{\partial{}^*C^{0}_{ab}(0)}{\partial y^{c}} &= \frac{n+1}{n-1}\,Q^{i}_{abci}, \\ \frac{\partial^2{}^*C^{0}_{ab}(0)}{\partial y^{c}\,\partial y^{d}} &= \frac{n+1}{n-1}\,(Q^{i}_{abcdi} - Q^{j}_{ibc}Q^{i}_{jad} - Q^{j}_{iac}Q^{i}_{jbd}), \\ \frac{\partial{}^*C^{0}_{abc}(0)}{\partial yd} &= \frac{1}{3}\underset{a,b,c}{P}\left(\frac{\partial^2{}^*C^{0}_{ab}(0)}{\partial y^{c}\,\partial y^{d}}\right), \end{aligned} \tag{6.15}$$

from which follow by (4.1), (4.2), and (4.3)

$$ {}^*C^{0}_{abcd}(0) = -\,\frac{1}{6}\cdot\frac{n+1}{n-1}\,(Q^{j}_{abi}Q^{i}_{cdj} + Q^{j}_{aci}Q^{i}_{bdj} + Q^{j}_{adi}Q^{i}_{bcj}). \tag{6.16}$$

By the use of (6.5), we obtain

$$ {}^*C^{i}_{abcde}(0) = \frac{1}{5}P\left(\frac{\partial{}^*C^{i}_{abcd}(0)}{\partial ye}\right), \tag{6.17}$$

which after a rather lengthy calculation reduces to

$$ {}^*C^{i}_{abcde}(0) = \frac{-\,2}{5(n+1)}\cdot P(\delta^{i}_{a}\,{}^*C^{0}_{bcde}(0)) \tag{6.18}$$

From (6.14) we now get

$$(6.19)\quad \begin{aligned} {}^*A^i_{abcde} &= Q^i_{abcde} + \frac{2}{5(n+1)} \cdot P(\delta^i_a {}^*C^0_{bcde}(0) \\ &+ \frac{1}{n+1}(\delta^i_a {}^*C^0_{bcde}(0) + \delta^i_b {}^*C^0_{acde}(0)) - \frac{1}{n+1} \underset{c,d,e}{S} (\delta^i_e {}^*A^0_{abcd}), \end{aligned}$$

and from (6.4)

$$(6.20)\quad \begin{aligned} {}^*A^0_{abcd} &= \frac{n+1}{n-1}(Q^i_{abcdi} - Q^j_{ibc} \cdot Q^i_{jad} - Q^i_{jac} Q^j_{ibd}) \\ &+ \frac{1}{6} \cdot \frac{n+1}{n-1}(Q^j_{abi} \cdot Q^i_{cdj} + Q^j_{aci} \cdot Q^i_{bdj} + Q^j_{adi} \cdot Q^i_{bcj}). \end{aligned}$$

Finally by (6.5) we calculate

$$(6.21)\quad \begin{gathered} {}^*C^0_{0bcd}(0) = 0, \qquad {}^*C^0_{00cd}(0) = 0, \qquad {}^*C^0_{000d}(0) = 0, \\ {}^*C^0_{0000}(0) = \frac{-6}{(n+1)^3}, \end{gathered}$$

and from these and (6.4)

$$(6.22)\quad \begin{gathered} {}^*A^0_{0bcd} = \frac{-1}{n-1} Q^i_{cdbi}, \qquad {}^*A^0_{00cd} = 0, \qquad {}^*A^0_{000d} = 0, \\ {}^*A^0_{0000} = 0, \qquad {}^*A^0_{a000} = 0, \qquad {}^*A^0_{ab00} = 0, \qquad {}^*A^0_{0b0d} = 0, \\ {}^*A^0_{abc0} = \frac{2}{n-1} Q^i_{abci}. \end{gathered}$$

Thus by (6.16), (6.19), (6.20), and (6.22) we have expressed the desired $^*A$ in terms of the Q's.

To obtain the relations between Q^i_{abcde} for $n = 2$, we select

$$(6.23)\quad Q^1_{11112} = A, \qquad Q^2_{22221} = B, \qquad Q^1_{11122} = C, \qquad Q^2_{22211} = D,$$

from which results

$$(6.24)\quad \begin{gathered} Q^1_{12111} = -\frac{3}{2}A, \qquad Q^2_{21222} = -\frac{3}{2}B, \qquad Q^2_{21112} = -A, \\ Q^2_{21122} = -C, \qquad Q^1_{12112} = -D, \qquad Q^1_{12122} = -B, \\ Q^2_{11222} = 6C - 3D, \qquad Q^1_{22111} = 6D - 3C, \qquad Q^2_{22111} = \frac{3}{2}A, \\ Q^1_{11222} = \frac{3}{2}B, \qquad Q^2_{11122} = \frac{3}{2}A, \qquad Q^1_{22211} = \frac{3}{2}B, \\ Q^2_{21111} = 0, \qquad Q^1_{12222} = 0, \qquad Q^2_{11112} = 0, \qquad Q^1_{22221} = 0, \\ Q^i_{11111} = 0, \qquad Q^i_{22222} = 0. \end{gathered}$$

From (6.19) and (6.20) we now obtain for $n = 2$

$$
\begin{aligned}
&{}^*A^i_{abcde} = Q^i_{abcde} - \frac{1}{3}(\delta^i_e {}^*A^0_{abcd} + \delta^i_d {}^*A^0_{abce} + \delta^i_c {}^*A^0_{abde}), \\
&{}^*A^0_{abcd} = 3 \cdot Q^i_{abcdi} .
\end{aligned} \tag{6.25}
$$

On substituting from (6.24) in (6.25) there results

$$
\begin{aligned}
&{}^*A^1_{11112} = -4A, \quad {}^*A^1_{12111} = 6A, \quad {}^*A^1_{12222} = 0, \\
&{}^*A^2_{11112} = 0, \quad {}^*A^2_{12111} = 0, \quad {}^*A^2_{12222} = 6B, \\
&{}^*A^1_{11122} = 3D - 6C, \quad {}^*A^1_{12112} = 2C + D, \quad {}^*A^1_{22111} = 6C - 15D, \\
&{}^*A^2_{11122} = -\frac{7}{2}A, \quad {}^*A^2_{12112} = \frac{3}{2}A, \quad {}^*A^2_{22111} = \frac{3}{2}A, \\
&{}^*A^1_{11222} = \frac{3}{2}B, \quad {}^*A^1_{12122} = \frac{3}{2}B, \quad {}^*A^1_{22112} = -\frac{7}{2}B, \\
&{}^*A^2_{11222} = 6D - 15C, \quad {}^*A^2_{12122} = 2D + C, \quad {}^*A^2_{22112} = 3C - 6D, \\
&{}^*A^1_{22122} = 0, \quad {}^*A^2_{22122} = -4B, \quad {}^*A^i_{11111} = {}^*A^i_{22222} = 0,
\end{aligned} \tag{6.26}
$$

and

$$
\begin{aligned}
&{}^*A^0_{1112} = \frac{15}{2}A, \quad {}^*A^0_{1122} = 21C - 9D, \quad {}^*A^0_{1211} = -\frac{15}{2}A, \\
&{}^*A^0_{1212} = -3C - 3D, \quad {}^*A^0_{1222} = -\frac{15}{2}B, \quad {}^*A^0_{2211} = 21D - 9C, \\
&{}^*A^0_{2212} = \frac{15}{2}B, \quad {}^*A^0_{1111} = 0, \quad {}^*A^0_{2222} = 0 .
\end{aligned} \tag{6.27}
$$

Finally from (6.22) we get

$$
\begin{aligned}
&{}^*A^0_{1110} = 0, \quad {}^*A^0_{2210} = -2\beta, \quad {}^*A^0_{01112} = \frac{-1}{2}\alpha, \\
&{}^*A^0_{1120} = -2\alpha, \quad {}^*A^0_{2220} = 0, \quad {}^*A^0_{0212} = \frac{-1}{2}\beta, \\
&{}^*A^0_{1210} = \alpha, \quad {}^*A^0_{0111} = 0, \quad {}^*A^0_{0122} = \beta, \\
&{}^*A^0_{1220} = \beta, \quad {}^*A^0_{0211} = \alpha, \quad {}^*A^0_{0222} = 0,
\end{aligned} \tag{6.28}
$$

where α, β are defined under (6.12).

On putting

$$
-\frac{15}{2}A = A', \quad -\frac{15}{2}B = B', \quad 21D - 9C = C', \quad 21C - 9D = D', \tag{6.29}
$$

we obtain from (2.1) the desired differential equations

$$
(6.30)\quad
\begin{array}{ll}
X_0^1: & \frac{5}{2}\alpha\frac{\partial}{\partial A'} - \frac{7}{2}\beta\frac{\partial}{\partial C'} + \frac{3}{2}\beta\frac{\partial}{\partial D'} = 0,\\[2mm]
X_1^1: & 2\alpha\frac{\partial}{\partial\alpha} + \beta\frac{\partial}{\partial\beta} + 3A'\frac{\partial}{\partial A'} + B'\frac{\partial}{\partial B'} + 2C'\frac{\partial}{\partial C'} + 2D'\frac{\partial}{\partial D'} = 0,\\[2mm]
X_2^1: & -\beta\frac{\partial}{\partial\alpha} + \frac{C'-D'}{2}\frac{\partial}{\partial A'} - 2B'\frac{\partial}{\partial C'} + 2B'\frac{\partial}{\partial D'} = 0,\\[2mm]
X_0^2: & \frac{5}{2}\beta\frac{\partial}{\partial B'} + \frac{3}{2}\alpha\frac{\partial}{\partial C'} - \frac{7}{2}\alpha\frac{\partial}{\partial D'} = 0,\\[2mm]
X_1^2: & -\alpha\frac{\partial}{\partial\beta} + \frac{D'-C'}{2}\frac{\partial}{\partial B'} + 2A'\frac{\partial}{\partial C'} - 2A'\frac{\partial}{\partial D'} = 0,\\[2mm]
X_2^2: & \alpha\frac{\partial}{\partial\alpha} + 2\beta\frac{\partial}{\partial\beta} + A'\frac{\partial}{\partial A'} + 3B'\frac{\partial}{\partial B'} + 2C'\frac{\partial}{\partial C'} + 2D'\frac{\partial}{\partial D'} = 0.
\end{array}
$$

The determinant of the coefficients is

$$
\begin{vmatrix}
0 & 0 & \frac{5}{2}\alpha & 0 & -\frac{7}{2}\beta & \frac{3}{2}\beta\\
2\alpha & \beta & 3A' & B' & 2C' & 2D'\\
-\beta & 0 & \frac{1}{2}(C'-D') & 0 & -2B' & 2B'\\
0 & 0 & 0 & \frac{5}{2}\beta & \frac{3}{2}\alpha & -\frac{7}{2}\alpha\\
0 & -\alpha & 0 & \frac{1}{2}(D'-C') & 2A' & -2A'\\
\alpha & 2\beta & A' & 3B' & 2C' & 2D'
\end{vmatrix}.
$$

Putting $\alpha = 0$ the determinant has the value $-50\beta^4A'^2$ which shows

[6.3] *For $n = 2$ there are no projective scalars of the third order.*

However, since as just shown, (6.30) are independent, we have from [3.1],

[6.4] *For $n = 2$ and $p > 3$ there exist projective scalars.*

7. **Number of independent scalars.** If we let $^*N(n, p)$ represent the number of functionally independent solutions of (2.1), then it follows that

$$
^*N(n, p) = \sum_{r=1}^{p} N(n, r) - n(n+1), \qquad \begin{pmatrix} n \geqq 3,\ p > 1,\\ n = 2,\ p \geqq 4, \end{pmatrix}
$$

$$
^*N(n, p) = N(n, 1) - n^2, \qquad (n \geqq 3,\ p = 1).
$$

Then $^*N(n, p)$ gives the number of projective scalars in a first fundamental set of order p, the proof of this being the same as in I, p. 217, except for $p = 2$. For this latter case we must show

$$
^*N(n, 2) - {}^*N(n, 1) > 0, \qquad (n \geqq 3).
$$

This reduces to showing

$$\frac{n}{24}(5n^4 + 6n^3 - 23n^2 - 12n - 24) > 0, \qquad (n \geqq 3). \tag{7.1}$$

By Descarte's rule of signs we see that the equation

$$f(x) = 5x^4 + 6x^3 - 23x^2 - 12x - 24 = 0,$$

has only one positive root, and this lies between $x = 2$ and $x = 3$. As $f(3) > 0$; it follows that (7.1) is true for the indicated values of n.

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PROJECTIONS IN ABSTRACT SPACES

By Charles C. Torrance

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1. It is well known that in the theory of linear operations the orthogonal projections play a fundamental rôle. In connection with these projections it is a basic theorem that every closed linear subspace of a given vector[1] space (supposed complete, with an inner product) determines an orthogonal projection. It is the object of this paper to generalize this theorem.

Definition 1. *If to each pair of elements f, g of a space S there corresponds a unique element $f + g$ of S, and if this correspondence is such that, for arbitrary elements f, g, h of S,*

1) $$f + g = g + f,$$

2) $$(f + g) + h = f + (g + h),$$

3) *the equation $g + x = f$ has at least one solution x in S, then S is called additive (or also an Abelian group).*

It is readily shown that equation 3 has a unique solution x in S so that the operation $+$ has a single-valued inverse, $-$, which obeys the usual rules of computation; also that there exists a unique element 0 in S such that $f + 0 = f$.

Definition 2. *Let S, C, and Σ be three sets of elements $f, g, \cdots, a, b, \cdots,$ and $\varphi, \psi, \cdots$ respectively; let $a\varphi$ and (f, φ) be two operations such that*

a) *Corresponding to each element a in C and to each element φ in Σ there is associated a unique element $f = a\varphi$ in S; conversely, there corresponds to each element f in S, other than a certain element f_0, a unique representation $f = a\varphi$, and there corresponds to each element φ in Σ a unique representation of f_0, $f_0 = a\varphi$,*

b) *Corresponding to each element f in S and to each element φ in Σ there is associated a unique element $a = (f, \varphi)$ in C,*

c) *For each element a in C and each element φ in Σ, $(a\varphi, \varphi) = a$.*

In the notation of the preceding definition, let φ' be the set of all elements $a\varphi$, where a ranges through C but where φ is constant. Then φ' is a subset of S. Let Σ' be the set of all sets $\varphi', \psi', \cdots$. The sets $\varphi', \psi', \cdots$ have the following properties:

1) All the sets $\varphi', \psi', \cdots$ in Σ' have a common element f_0, but are otherwise pairwise disjoint,

2) If f is an element of S, at least one of the sets φ' in Σ' contains it,

3) The elements of each set φ' in Σ' are in biunique correspondence with the elements of C,

[1] Banach, Fund. Math., 3 (1922) p. 134; Hahn Monatshefte, 32 (1922) p. 1.

4) Corresponding to each element f in S and to each set φ' in Σ' there is associated a unique element $f_{\varphi'}$ in φ', the element $f_{\varphi'}$ being f itself in the case where φ' contains f; $f_{\varphi'}$ will be called the projection of f upon φ'.

Properties 1, 2, and 3 result from part a of Definition 2, and property 4 results from parts b and c, where $f_{\varphi'} = (f, \varphi)\,\varphi$.

It will now be shown that properties 1, 2, 3, and 4 are sufficient to define the operations $a\varphi$ and (f, φ). Let S and C be two sets of elements $f, g, \cdots$ and $a, b, \cdots$; let Σ' be a set of subsets $\varphi', \psi', \cdots$ in S having properties 1, 2, 3, and 4. Then

a′) Corresponding to each element a in C and to each set φ' in Σ' there is associated by 3 a unique element $f = a\varphi'$ in φ'; conversely, there corresponds by 1, 2, 3 to each element $f \neq f_0$ in S a unique representation $f = a\varphi'$, where φ' contains f, and there corresponds to each set φ' in Σ' a unique representation of f_0, $f_0 = a\varphi'$,

b′) Corresponding to each element f in S and to each set φ' in Σ' there is associated by 4 a unique element $f_{\varphi'}$ in φ' and with $f_{\varphi'}$ in turn there is associated by 3 a unique element $a = (f, \varphi')$ in C,

c′) For each element a in C and each set φ' in Σ', $(a\varphi', \varphi') = a$ (by 4, a′, and b′).

It follows further from these conditions that $f_{\varphi'} = (f, \varphi')\varphi'$ and that, if φ' contains f, $f = (f, \varphi')\,\varphi'$.

These conditions are merely conditions a, b, and c with φ replaced by φ' and Σ replaced by Σ'. Since the sets φ' and the elements φ are in biunique correspondence, the substitution is immaterial. In what follows, no distinction will be made between φ and φ'.

2. The following definition and postulates are introduced.

POSTULATE A. *The spaces C and S are additive.*

DEFINITION 3. *If (f, φ) is the function defined above, and if, for arbitrary elements f, g in S and arbitrary φ in Σ, $(f + g, \varphi) = (f, \varphi) + (g, \varphi)$, then (f, φ) is called left-additive.*

POSTULATE B. *The function (f, φ) defined above is left-additive.*

POSTULATE C. *If φ and ψ are in Σ and if a is an element of C such that $(a\varphi, \psi) = 0$ with $a\varphi \neq f_0$, then $(b\varphi, \psi) = 0$ and $(b\psi, \varphi) = 0$ for every element b in C.*

THEOREM 1. *For each φ in Σ: first, $(0, \varphi) = 0$, and second, $a\varphi = 0$ when and only when $a = 0$; $f_0 = 0$.*

By Postulate B, $(g, \varphi) = (g + f + (-f), \varphi) = (g, \varphi) + (f, \varphi) + (-f, \varphi)$ for arbitrary f, g, and φ; hence $(f, \varphi) + (-f, \varphi) = 0$ and $(-f, \varphi) = -(f, \varphi)$. It follows that $(0, \varphi) = (f - f, \varphi) = (f, \varphi) - (f, \varphi) = 0$. Suppose it were the case that $0 \neq f_0$; let $0 = a_0\varphi_0$. Since $(a_0\varphi_0, \varphi_0) = 0$, then, by Postulate C, $(a\varphi_0, \varphi_0) = 0$ for every element a in C. But by condition C above it follows that $(a\varphi_0, \varphi_0) = 0$ only when $a = 0$. Hence $0 = f_0$. Let φ be arbitrary in Σ, and let $0 = a\varphi$. Then $(a\varphi, \varphi) = 0$ and, by condition C, $a = 0$.

DEFINITION 4. *The relation of orthogonality between two elements in S is sup-*

posed defined so that a sufficient condition that f be orthogonal to $g = b\psi$ is that $(f, \psi) = 0$, and also that this condition is necessary and sufficient that f be orthogonal to every element in ψ, in which case f is said to be orthogonal to ψ itself.

REMARK 1. It follows from this definition that if f is orthogonal to ψ it is orthogonal to each element in ψ. By Theorem 1, the element 0 is orthogonal to every element in S; it will be apparent from what follows that, if there are two distinct spaces (elements) in Σ, every element in S is orthogonal to 0. If a, φ, ψ are such that $(a\varphi, \psi) = 0$ with $a\varphi \neq 0$, then, by Postulate C and condition c above, $\varphi \neq \psi$, each element of φ is orthogonal to (each element of) ψ, and each element of ψ is orthogonal to (each element of) φ; in this circumstance the spaces φ and ψ are called orthogonal.

THEOREM 2. *If f is an arbitrary element of S and if φ is an arbitrary space in Σ, then f has a unique representation of the form $f = f_1 + f_2$, where f_1 is in φ and f_2 is orthogonal to φ; in this representation $f_1 = (f, \varphi)\,\varphi$ and $f_2 = (f, \psi)\,\psi$, where ψ is in Σ, contains $f - f_1$, and is orthogonal to φ.*

Suppose first that φ does not contain f. Let $f - (f, \varphi)\varphi = b\psi$. Since $b\psi \neq 0$ and since $(b\psi, \varphi) = (f - (f, \varphi)\varphi, \varphi) = (f, \varphi) - (f, \varphi) = 0$, φ and ψ are orthogonal and the representation $f = (f, \varphi)\varphi + b\psi$ satisfies the condition in the theorem. To show that this representation is unique, suppose that $f = a\varphi + c\theta$, where $c\theta$ (and hence θ) is orthogonal to φ. Then $(f, \varphi) = a$ and $(f, \theta) = c$, so that a, and hence the element $c\theta$, are uniquely determined. It is evident that if φ contains f, the representation $f = f + 0$ is unique.

THEOREM 3. *The spaces φ in Σ are additive.*

Let a, b, and φ be arbitrary elements in C and Σ, and let $a\varphi + b\varphi = c\varphi + d\psi$, where ψ is orthogonal to φ. (Such a representation exists by Theorem 2.) By Postulate B, $(a\varphi + b\varphi, \varphi) = (c\varphi + d\psi, \varphi)$ and $c = a + b$; likewise $(a\varphi + b\varphi, \psi) = (c\varphi + d\psi, \psi)$ and $d = 0$. It is evident that $(-a)\varphi = -(a\varphi)$ and the theorem follows immediately from Definition 1.

DEFINITION 5. *If Δ is a subset of Σ such that each pair of spaces in Δ are orthogonal, then Δ is called an orthogonal set. Let T be an arbitrary subset of S; let each element $f \neq 0$ in T be represented as $f = a\varphi$; let τ be the set of spaces φ appearing in this set of representations; τ is called the set of spaces φ in T. An orthogonal subset Δ of τ is said to be complete in T if there exists no space ψ in $\tau - \Delta$ such that ψ together with Δ form an orthogonal set.*

THEOREM 4. *A necessary and sufficient condition that an orthogonal subset Δ of Σ be complete in a subset T of S is that there exists no element $f \neq 0$ in T such that $(f, \varphi) = 0$ for all spaces φ in Δ.*

The necessity of this condition follows from the last part of Remark 1; the sufficiency follows from Definition 5.

THEOREM 5. *There exists a complete orthogonal set Δ in every subset T of S.*

Let the elements of T other than 0 be well-ordered so that to each element $f \neq 0$ of T there is attached an ordinal number α. Let $f_\alpha = a_\alpha\varphi_\alpha$. The elements ψ_β of Δ will be selected from among the elements φ_α by transfinite induction. Suppose that all ψ_β, $\beta < \alpha$, have already been selected. If $(a_\alpha\varphi_\alpha, \psi_\beta) = 0$ for

every $\beta < \alpha$ for which ψ_β is defined, then ψ_α is taken to be φ_α; otherwise ψ_α is undefined. It is evident that the set Δ so defined is orthogonal. Let τ be the set of all spaces φ_α. Suppose there exists an element φ in τ such that φ is orthogonal to Δ. Then φ has a smallest ordinal number α, $\varphi = \varphi_\alpha$, and $(a_\alpha \varphi_\alpha, \psi_\beta) = 0$ for every $\beta < \alpha$ for which ψ_β is defined. Hence φ_α was included in Δ, and Δ is complete.

DEFINITION 6. *An orthogonal set Δ is called closed if, for each element f in S, there is associated an element f_Δ in S such that* 1) $(f_\Delta, \varphi) = (f, \varphi)$ *for each element φ in Δ, and* 2) $(f_\Delta, \varphi) = 0$ *for each φ orthogonal to Δ (that is, to each element of Δ). If a closed orthogonal set Δ is such that, for all f in S, f_Δ is in a subset M of S, then Δ is called M-contained.*

Suppose there were two elements f_Δ and f'_Δ satisfying the conditions of the preceding definition. Then, for each φ in Δ and for each φ orthogonal to Δ, $(f_\Delta - f'_\Delta, \varphi) = (f, \varphi) - (f, \varphi) = 0$. Since there exists a complete orthogonal set $\Delta' \supset \Delta$ in the entire space S, $f_\Delta - f'_\Delta = 0$ by Theorem 4. It also follows from Theorem 4 that if f is orthogonal to Δ, then $f_\Delta = 0$ (see Theorem 7). Every finite set Δ is closed and $f_\Delta = \sum_{\varphi \epsilon \Delta} (f, \varphi)\varphi$.

3. For the sake of interest, sufficient conditions will now be given that every orthogonal set Δ be closed. Insofar as is possible, the additional assumptions needed will be imposed upon C.

Let (a, b) be a single-valued function defined over all ordered pairs of elements in C with values in an additive space N.

(α) It is assumed that, for arbitrary elements a, b, c of C, $(a + b, c) = (a, c) + (b, c)$ and $(a, b + c) = (a, b) + (a, c)$, i.e., that the function (a, b) is "biadditive"; also that $(a, a) = 0$ only when $a = 0$.

REMARK 2. For every element a in C, $(0, a) = (a, 0) = 0$. The proof of this statement is like that of the first part of Theorem 1.

(β) It is assumed that, for arbitrary elements a, b, φ, and ψ, $((a\varphi, \psi), b) = (a, (b\psi, \varphi))$.

If $f = a\varphi$ and $g = b\psi$ are arbitrary elements of S, then (f, g) is defined by the condition $(f, g) = (a\varphi, b\psi) = ((a\varphi, \psi), b) = (a, (b\psi, \varphi))$. Inasmuch as the element 0 has various representations, it is *a priori* possible that this definition might lead to inconsistencies. But, by virtue of Theorem 1 and Remark 2, there can exist no such inconsistency.

REMARK 3. The function (f, g) is biadditive. This follows from the fact that if $f = a\varphi$, $g = b\psi$, and $h = c\theta$ are arbitrary elements of S, then $(f + g, h) = (a\varphi + b\psi, c\theta) = ((a\varphi + b\psi, \theta), c) = ((a\varphi, \theta) + (b\psi, \theta), c) = ((a\varphi, \theta), c) + ((b\psi, \theta), c) = (a\varphi, c\theta) + (b\psi, c\theta) = (f, h) + (g, h)$. The remainder of the proof is similar to the part given. From (α) it follows that $(f, f) = 0$ only when $f = 0$.

REMARK 4. If f is said to be orthogonal to g when $(f, g) = 0$, then the condition in Definition 4 remains sufficient, but is not necessary. It is entirely possible for two elements in a space φ to be orthogonal and distinct from 0.

Let $p, q, \cdots$ denote arbitrary elements of the space N (where N contains the range of the function (a, b)).

(γ) It is assumed that the symbol $\lim_{n\to\infty} p_n$ is defined and single-valued over a subset of the sequences $\{p_n\}$ in N, that $\lim_{m,n\to\infty} p_{mn}$ is analogously defined, and that these symbols are subject to the usual theorems concerning limits.

By $\lim_{n\to\infty} f_n = f$ it is meant that $\lim_{n\to\infty} (f_n - f, f_n - f) = 0$. The function (f, g) is said to be continuous at f', g' if $\lim_{m,n\to\infty} (f_m, g_n) = (f', g')$ for every pair of sequences $\{f_m\}$ and $\{g_n\}$ with limits f' and g' respectively.

(δ) It is assumed that (f, g) is continuous throughout S.

From this assumption and Remark 3 (last part) it is possible to prove that $\lim_{n\to\infty} f_n$, if it exists, is unique.

(ϵ) It is assumed that the set of all real numbers is included in N, with the usual meaning of addition and convergence. For each element a in C, (a, a) (and hence (f, f)) is a non-negative real number.

Since $(f + g, f + g) = (f, f) + (g, g) + (f, g) + (g, f)$, it follows from this assumption that $(f, g) + (g, f)$ is real; other than this, no restriction is made on (f, g) when f and g are distinct.

Bessel's inequality. Let $\varphi_1, \varphi_2, \cdots, \varphi_n$ be a finite orthogonal set Δ. Then

$$\left(f - \sum_{i=1}^{n} a_i\varphi_i, f - \sum_{i=1}^{n} a_i\varphi_i\right)$$

$$= (f,f) - \sum_{i=1}^{n} (f, a_i\varphi_i) - \sum_{i=1}^{n} (a_i\varphi_i, f) + \sum_{i,j=1}^{n} (a_i\varphi_i, a_j\varphi_j)$$

$$= (f,f) - \sum_{i=1}^{n} ((f, \varphi_i), a_i) - \sum_{i=1}^{n} (a_i, (f, \varphi_i)) + \sum_{i=1}^{n} (a_i, a_i)$$

$$= \sum_{i=1}^{n} [(a_i, a_i) - ((f, \varphi_i), a_i) - (a_i, (f, \varphi_i)) + ((f, \varphi_i), (f, \varphi_i))]$$

$$+ (f,f) - \sum_{i=1}^{n} ((f, \varphi_i), (f, \varphi_i))$$

$$= \sum_{i=1}^{n} (a_i - (f, \varphi_i), a_i - (f, \varphi_i)) \quad + (f,f) - \sum_{i=1}^{n} ((f, \varphi_i), (f, \varphi_i)).$$

The left member of this equation is minimized when $a_i = (f, \varphi_i)$. Since the left member is non-negative,

$$\sum_{i=1}^{n} ((f, \varphi_i), (f, \varphi_i)) \leqq (f,f).$$

It follows from this inequality that, if Δ is a countable set of orthogonal spaces φ_i, then $\sum_{i=1}^{\infty} ((f, \varphi_i), (f, \varphi_i))$ is convergent and not greater than (f, f). Likewise, if Δ is non-countable, then $((f, \varphi_i), (f, \varphi_i)) = 0$ and $(f, \varphi_i) = 0$ for all but a countable subset of Δ. This leads to

REMARK 5. For any orthogonal set Δ, the expresssion $\sum_{\varphi\epsilon\Delta}(f, \varphi)\varphi$ has at most a countable set of non-null terms and $\Sigma_{\varphi\epsilon\Delta}((f, \varphi), (f, \varphi))$ is convergent.

REMARK 6. Let $\sum_{i=1}^{\infty} f_i$ be defined as $\lim_{n\to\infty} \sum_{i=1}^{n} f_i$ (provided this limit exists). If $\varphi_1, \varphi_2, \cdots$ is a countable orthogonal set Δ, and if $f = \sum_{i=1}^{\infty} a_i\varphi_i$, then $a_j = (f, \varphi_j)$ since (f, g) is continuous and since $a_j = \lim_{n\to\infty} (\sum_{i=1}^{n} a_i\varphi_i, \varphi_j) = (\sum_{i=1}^{\infty} a_i\varphi_i, \varphi_j)$; if ψ is orthogonal to Δ, $(f, \psi) = 0$.

A sequence $f_1, f_2, \cdots$ is called fundamental if $\lim_{m,n\to\infty} (f_m - f_n, f_m - f_n) = 0$. It is evident that $0 \leqq (f - g, f - g) = (f, f) + (g, g) - (f, g) - (g, f)$, so that $(f, g) + (g, f) \leqq (f, f) + (g, g)$. Hence $(f + g, f + g) \leqq 2\{(f, f) + (g, g)\}$, and $(f_m - f_n, f_m - f_n) = ((f_m - f) + (f - f_n), (f_m - f) + (f - f_n)) \leqq 2\{(f_m - f, f_m - f) + (f - f_n, f - f_n)\}$. It follows that $\lim_{m,n\to\infty} (f_m - f_n) = 0$ when $\lim_{n\to\infty} f_n = f$, that is, a convergent sequence is fundamental. If S is such that every fundamental sequence has a limit, then S is called complete.

(η) It is assumed that S is complete.

It may be mentioned that S can always be extended to be complete if it is not already so.

REMARK 7. If $\varphi_1, \varphi_2, \cdots$ is a countable orthogonal set, then a necessary and sufficient condition that $\sum_{i=1}^{\infty} a_i\varphi_i$ be convergent is that $\sum_{i=1}^{\infty} (a_i, a_i)$ be convergent. This follows from the fact that $\sum_{i=1}^{\infty} a_i\varphi_i$ is convergent when and only when the sequence $\sum_{i=1}^{n} a_i \varphi_i$ is fundamental, and that $\lim_{m,n\to\infty} (\sum_{i=1}^{m} a_i\varphi_i - \sum_{i=1}^{n} a_i\varphi_i, \sum_{i=1}^{m} a_i\varphi_i - \sum_{i=1}^{n} a_i\varphi_i) = \lim_{m,n\to\infty} (\sum_i a_i\varphi_i, \sum_i a_i\varphi_i) = \lim_{m,n\to\infty} (a_i, \varphi_i)$ whenever either the first or last limit exists, where the symbol $\sum_i$ denotes either $\sum_{i=n+1}^{m}$ or $\sum_{i=m+1}^{n}$ or 0 according as $m > n$, $m < n$, or $m = n$.

Under the assumptions listed above, the closure of an orthogonal set Δ follows from Remarks 5, 6, and 7.

4. In the remainder of this discussion no use is made of assumptions α to η above.

THEOREM 6. *If f is an arbitrary element in S and if Δ is an arbitrary closed orthogonal set in Σ, then $f - f_\Delta$ is orthogonal to Δ.*

This follows immediately from Postulate B and Definition 6.

DEFINITION 7. *A subspace M of S is called a projection space if* 1) *M is additive, and* 2) *at least one of the complete orthogonal sets Δ in M is M-contained.*

THEOREM 7. *If M is an additive space in S, a necessary and sufficient condition that an M-contained orthogonal set Δ in M be complete in M is that, for each element f of M, $f = f_\Delta$.*

As to the necessity of the condition: for each element f in M, f_Δ is in M (Definition 6), $f - f_\Delta$ is in M (Definition 1), and $f - f_\Delta$ is orthogonal to Δ (Theorem 6). Since Δ is complete in M, $f - f_\Delta = 0$ (Theorem 4). As to the sufficiency of the condition: suppose $f = f_\Delta$ to be in M and orthogonal to Δ. Then $(f, \varphi) = 0$ for every φ in Δ and for every φ orthogonal to Δ (Definition 6). Hence $f = 0$ (Theorem 4) and Δ is complete in M (Theorem 4).

THEOREM 8. *If f is an arbitrary element of S and if M is an arbitrary projection space in S, then f has a unique representation of the form $f = f_1 + f_2$, where f_1 is in M and f_2 is orthogonal to M and in S.*

By Definition 7 there exists an M-contained complete orthogonal set Δ in M. Let $f_1 = f_\Delta$ and let $f_2 = f - f_1$. Then f_1 is in M, and by Theorem 6, f_2 is orthogonal to Δ. If $f_2 = 0$, f_2 is orthogonal to M and the representation satisfies the condition in the theorem. If $f_2 \neq 0$, then f_2 has a unique representation of the from $f_2 = a\psi$, and ψ and Δ form an orthogonal set. Since Δ is complete in M, it follows by Theorem 7 that an arbitrary element g of M is representable as $g = g_\Delta$. By Definition 6, $(g, \psi) = 0$. Hence ψ is orthogonal to each space θ in M (in the sense of Definition 5), and f_2 is orthogonal to M. Thus the representation $f = f_1 + f_2$ satisfies the condition in the theorem. To show that this representation is unique, suppose there were two distinct such representations, $f = f_1 + f_2 = f_1' + f_2'$. Then $f_1 - f_1' = f_2' - f_2 \neq 0$. Since M is additive, $f_1 - f_1'$ is in M, so that $f_1 - f_1' = b\varphi$, where φ is in M. By Postulate B, $(f_2' - f_2, \theta) = 0$ for each space θ in M; this holds in particular for the space φ, so that $(f_2' - f_2, \varphi) = (b\varphi, \varphi) = 0$. Hence $b = 0$ and the representation is unique.

5. It may be of interest to close this discussion by proposing some questions. Suppose a linear subspace L of S is characterized by the property that if $f = a\varphi \neq 0$ and $g = b\psi \neq 0$ are two elements of L, then $a\varphi + b\psi$ is in L for all a and b in C. Is a projection space M linear? To answer this question affirmatively it is sufficient, by virtue of condition 1 in Definition 7, to show that if $a\varphi \neq 0$ is in M, then each element of φ is in M. It is evident that the answer to this question is affirmative in the case of ordinary vector spaces. Related to this question, and perhaps fundamental to it, is the following: if $a\varphi = (a\varphi, \psi)\,\psi + (a\varphi, \theta_1)\theta_1$ and $b\varphi = (b\varphi, \psi)\,\psi + (b\varphi, \theta_2)\,\theta_2$, then is it possible for θ_1 and θ_2 to be distinct when $a \neq 0$, $b \neq 0$, $a \neq b$, and $\varphi \neq \psi$?

CASE SCHOOL OF APPLIED SCIENCE.

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ON SEVERAL FAMILIES OF PLANE CURVES WITH AN APPLICATION TO DIFFERENTIAL EQUATIONS

By G. Van der Lyn*

(Received April 28, 1936)

1. Let D be a domain bounded by a closed Jordan curve J. Let us assume that a family F of simple continuous curves is defined in the domain, with the conditions that

1: Through every point of D and of its boundary passes one and only one curve of the family. (The curve through the point P will be designated by $C(P)$.)

2: No curve of the family has an end point in the interior of D.

Theorem. *The curve $C(P)$, regarded as a function of P, is continuous; that is to say, if P_n is a sequence of points converging to the point P, the curves $C(P_n)$ converge uniformly to the curve $C(P)$.*

Proof. Let us suppose our theorem false. Then, there exists a positive number ϵ and a sequence P_{n_i} contained in the sequence P_n, such that each curve $C(P_{n_i})$ has a point p_i whose distance from the curve $C(P)$ is greater than ϵ. Let p be a limit point of the set p_i. This point p is not on the curve $C(P)$. From the sequence $C(P_{n_i})$ we can extract a new sequence such that the corresponding points p_i converge to p. Then, successively extracting new sequences of curves from this sequence, each sequence contained in the preceding, and using the well known "method of the diagonal", we can obtain a sequence of curves C_n^* which possesses the following property:

If E is the set of the points x such that every neighborhood of x has points in common with an infinite number of the curves C_n^*, then, if δ is a given neighborhood of x, there exists a number N such that every curve C_n^*, with $n > N$, has points in δ.

The set E is obviously a bounded continuum joining P and p.

Now, let us divide the points of E into disjoint sets E_μ in such a manner that two points of E belong to E_μ if and only if they belong to a same curve of the family F. Every set E_μ, being the intersection of two closed sets (the set E and a curve of the family F), is closed. As is well known,[1] a bounded continuum cannot be divided into a finite number (greater than 1), or into a countable number of closed disjoint sets. Therefore, if we prove that the sets E_μ are not uncountable, it will follow that there exists only one set E_μ. Then the set E would be an arc of a curve C_1 of the family F, joining P and p, and consequently distinct from the curve $C(P)$ which does not contain the point p. But this is impossible, since $C(P)$ is the only curve of the family F that passes through P.

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[1] Theorem of W. Sierpinski, Tôhoku Math. Jour. 13 (1918) p. 300.

It remains to prove that the sets E_μ cannot be uncountable. Let C_μ be the curve of the family F which contains E_μ. This curve has its end points A_μ and B_μ on the boundary J of the domain D. It divides this domain into two domains D_1 and D_2. The set E cannot have points in both those domains, for if E did contain a point p_1 in D_1 and a point p_2 in D_2, there would exist a curve of the family F passing sufficiently near the points p_1 and p_2 to have points in the domains D_1 and D_2; this curve would then have points in common with C_μ, contradicting our hypothesis.

Let $(A_\mu B_\mu)$ be the arc which is on the boundary of the domain that does not contain E. The arcs (A_μ, B_μ) corresponding to distinct curves C_μ cannot have points in common and therefore they are countable. Thus the proof of our theorem is achieved.

As is easily seen, the domain D may be supposed to be a non-simply connected domain, bounded by a finite number of simple Jordan curves, since such a domain can be divided into a finite number of simply connected domains by a few continuous curves. Moreover, the theorem still remains if the family F contains also closed curves, or curves with multiple points, provided that each curve contains a simple arc.

2. Let us consider the differential equation

$$y' = f(x, y), \tag{1}$$

where $f(x, y)$ is a finite function defined in a domain D such as in §1. Concerning the function $f(x, y)$ we suppose only that it is such that through every point of the domain D there passes one and only one solution of the equation (1), this solution being such that it can be extended to the boundary of D.

Then, *the function $f(x, y)$ belongs to the first class of Baire.*

Indeed, let $y = \varphi(x, x_0, y_0)$ be the solution of the equation (1) that passes through the point (x_0, y_0) of the domain D. If h_n is a sequence of numbers converging to zero, we have the equations

$$f(x_0, y_0) = \left[\frac{\partial\varphi(x, x_0, y_0)}{\partial x}\right]_{x=x_0} = \lim_{n\to\infty} \frac{\varphi(x_0 + h_n, x_0, y_0) - y_0}{h_n}.$$

The function $\varphi(x_0 + h_n, x_0, y_0)$ where h_n is a constant, is continuous in $(x_0\, y_0)$, as a result of the theorem in §1. Hence the ratios $(1/h_n)[\varphi(x_0 + h_n, x_0, y_0) - y_0]$ are continuous in $(x_0\, y_0)$ and their limit, $f(x_0, y_0)$, belongs to the first class of Baire.

3. It may be interesting to notice that this last theorem is false in a 3-dimensional space. If

$$y' = f(x, y, z)$$

$$z' = g(x, y, z)$$

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It remains to prove that the sets E_μ cannot be uncountable. Let C_μ be the curve of the family F which contains E_μ. This curve has its end points A_μ and B_μ on the boundary J of the domain D. It divides this domain into two domains D_1 and D_2. The set E cannot have points in both those domains, for if E did contain a point p_1 in D_1 and a point p_2 in D_2, there would exist a curve of the family F passing sufficiently near the points p_1 and p_2 to have points in the domains D_1 and D_2; this curve would then have points in common with C_μ, contradicting our hypothesis.

Let $(A_\mu B_\mu)$ be the arc which is on the boundary of the domain that does not contain E. The arcs (A_μ, B_μ) corresponding to distinct curves C_μ cannot have points in common and therefore they are countable. Thus the proof of our theorem is achieved.

As is easily seen, the domain D may be supposed to be a non-simply connected domain, bounded by a finite number of simple Jordan curves, since such a domain can be divided into a finite number of simply connected domains by a few continuous curves. Moreover, the theorem still remains if the family F contains also closed curves, or curves with multiple points, provided that each curve contains a simple arc.

2. Let us consider the differential equation

$$y' = f(x, y), \tag{1}$$

where $f(x, y)$ is a finite function defined in a domain D such as in §1. Concerning the function $f(x, y)$ we suppose only that it is such that through every point of the domain D there passes one and only one solution of the equation (1), this solution being such that it can be extended to the boundary of D.

Then, *the function $f(x, y)$ belongs to the first class of Baire.*

Indeed, let $y = \varphi(x, x_0, y_0)$ be the solution of the equation (1) that passes through the point (x_0, y_0) of the domain D. If h_n is a sequence of numbers converging to zero, we have the equations

$$f(x_0, y_0) = \left[\frac{\partial\varphi(x, x_0, y_0)}{\partial x}\right]_{x=x_0} = \lim_{n\to\infty} \frac{\varphi(x_0 + h_n, x_0, y_0) - y_0}{h_n}.$$

The function $\varphi(x_0 + h_n, x_0, y_0)$ where h_n is a constant, is continuous in $(x_0\, y_0)$, as a result of the theorem in §1. Hence the ratios $(1/h_n)[\varphi(x_0 + h_n, x_0, y_0) - y_0]$ are continuous in $(x_0\, y_0)$ and their limit, $f(x_0, y_0)$, belongs to the first class of Baire.

3. It may be interesting to notice that this last theorem is false in a 3-dimensional space. If

$$y' = f(x, y, z)$$

$$z' = g(x, y, z)$$

is a system of differential equations such that through every point passes one and only one integral curve, the function $f(x, y, z)$ and $g(x, y, z)$ may even be *not measurable*. This is shown by the following simple example

$$y' = f(z)$$

$$z' = 0$$

when $f(z)$ is an arbitrary finite function of z, measurable or not.

PRINCETON, NEW JERSEY.

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DIFFERENTIABLE MANIFOLDS[1]

By Hassler Whitney

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Introduction

The main purpose of this paper is to provide tools of a purely analytic character for a general study of the topology of differentiable manifolds, and maps of them into other manifolds. A differentiable manifold is generally defined in one of two ways; as a point set with neighborhoods homeomorphic with Euclidean space E_n, coördinates in overlapping neighborhoods being related by a differentiable transformation,[2] or as a subset of E_n, defined near each point by expressing some of the coördinates in terms of the others by differentiable functions.[3]

The first fundamental theorem is that the first definition is no more general than the second; any differentiable manifold may be imbedded in Euclidean space. In fact, it may be made into an analytic manifold in some E_n. As a corollary, it may be given an analytic Riemannian metric. The second fundamental theorem (when combined with the first) deals with the smoothing out of a manifold. Let f be a map of any character (continuous or differentiable, without an inverse) of a differentiable manifold M of dimension m into another, N, of dimension n. (Either manifold might be an open subset of Euclidean space.) Then if $n \geqq 2m$, we may alter f as little as we please, forming a regular map F. (A map is *regular* if, near each point, it is differentiable and has a differentiable inverse.) Moreover, if $n \geqq 2m + 1$, F may be made (1-1). We show in Theorem 6 that if $n \geqq 2m + 2$, then any two regular maps f_0, f_1 of M into E_n are equivalent, in the following sense. $f_0(M)$ may be deformed into $f_1(M)$ by maps $f_t(0 \leqq t \leqq 1)$ so that the path crossed by the manifold is the regular map of an $(m + 1)$-dimensional manifold. Moreover, if $n \geqq 2m + 3$, and $f_0(M)$ and $f_1(M)$ are non-singular, so is the $(m + 1)$-manifold.

A fundamental unsolved problem is the following: *Can any analytic manifold be mapped in an analytic manner into Euclidean space?*[4]

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[1] Presented to the Am. Math Soc. Sept. 1935. An outline of the paper will be found in Proc. Nat. Ac. of Sci., vol. 21 (1935), pp. 462–463.

[2] Differentiable manifolds have been studied for instance by O. Veblen and J. H. C. Whitehead, *The foundations of differential geometry*, Cambridge Tracts, 1932. An example of a differentiable (in fact, analytic) manifold is the manifold of k-planes through a point in n-space. See §24.

[3] Manifolds in E_n which are defined by the vanishing of a set of differentiable functions are of a special character; see H. Whitney, *The imbedding of manifolds* $\cdots$, in the October 1936 issue of these Annals.

[4] This seems quite probable. It is proved for some special analytic manifolds in §§23–24.

Theorem 1 shows only that there is a differentiable map (with all derivatives), such that the resulting point set forms an analytic manifold.

Many portions of the proofs are based on the Weierstrass approximation theorem, if the manifolds are closed; if they are open, this theorem must be replaced by a corresponding theorem on functions defined in open sets. This and other theorems which will be useful may be found in a previous paper.[5] In proving both fundamental theorems, the following method is used continually. Let f be a differentiable map of M into E_n, and let U be a small portion of M. We consider a class S of maps f' of U into E_n which approximate to f in U; S forms a part of a Euclidean space. The maps f' we do not wish are characterized by subsets of S whose dimensions may be learned,—we use here the notion of "k-extent" of a set similar to a definition of Carathéodory.[6] We find a desirable map f' in U, and do the same in other neighborhoods until we have found F in the whole manifold M.

The arrangement of the paper is indicated by the sentences introductory to each part.

CONTENTS

I. Definitions and Preliminary Results

In this part we collect definitions and facts which will be used constantly in what follows. In one section, §4, we assume a knowledge of the imbedding theorem and of Lemma 23.

1. Manifolds of class C^r. By an *m-dimensional manifold M of class C^r*, or, a *C^r-m-manifold* (r finite or $r = \infty$),[7] we shall mean a system composed of a set of points, which we shall also call M, and certain maps, as follows: Let $Q = Q_m$ be the interior of the unit $(m-1)$-sphere in Euclidean space E_m.[8] Let $\theta_1, \theta_2, \cdots$ be a finite or denumerable number of (1-1) maps of Q into M. Define the sets of points

$$(1.1)\qquad U_i = \theta_i(Q), \qquad U_{ij} = U_{ji} = U_i \cdot U_j, \qquad Q_{ij} = \theta_i^{-1}(U_{ij}),$$

[5] H. Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Trans. Am. Math. Soc., vol. 36 (1934), pp. 63–89. We refer to this paper as AE.

[6] C. Carathéodory, *Über das lineare Mass von Punktmengen*, Gött. Nachr., 1914, p. 426.

[7] We always suppose r is finite and > 0 unless otherwise stated. However, the results of §§1–4 all hold for $r = \infty$.

[8] That is, the space of all ordered sets of n real numbers.

and the (1-1) maps

(1.2) $$h_{ij}(x) = \theta_j^{-1}(\theta_i(x)) \text{ in } Q_{ij}.$$

We make four assumptions:

(α) The maps θ_i cover M: For each p in M there is an i and an x in Q such that $\theta_i(x) = p$.

(β) The Q_{ij} are open.

(γ)[9] There are no i, j, and sequence of points $\{x^k\}$ such that

$$x^k \text{ is in } Q_{ij}, \qquad x^k \to x \text{ in } Q - Q_{ij}, \qquad h_{ij}(x^k) \to x' \text{ in } Q - Q_{ji}.$$

(δ) $h_{ij}(x)$ (if defined) is of class C^r (see §2), and if $r > 0$, it has a non-vanishing Jacobian.

If, further, the h_{ij} are analytic, we say M is *analytic*. A 0-manifold consists of a finite or denumerable number of isolated points. A k-manifold ($k < 0$) contains no points.

We call the θ_i the *maps defining* M, and the U_i, *neighborhoods* in M. Note that M *need not be connected.*

We define *limit points* in M as follows: $p_k \to p$ if and only if there are an i, an s, a point x of Q and a sequence $\{x^k\}$ of points of Q such that

(1.3) $$\theta_i(x^k) = p_{s+k}, \qquad \theta_i(x) = p, \qquad x^k \to x.$$

We prove two facts. *If $p_k \to p$ and p is in U_j, then for some s, p_{s+k} is in U_j and $\theta_j^{-1}(p_{s+k}) \to \theta_j^{-1}(p)$.* For, say (1.3) holds. Then p is in U_{ij}, and hence $x = \theta_i^{-1}(p)$ is in Q_{ij}. As Q_{ij} is open and $x^k \to x$, there is an s such that x^{s+k} is in Q_{ij}. Hence $h_{ij}(x^{s+k}) = \theta_j^{-1}(p_{s+k})$ is in Q_{ji}, and as h_{ij} is continuous, $\theta_j^{-1}(p_{s+k}) \to \theta_j^{-1}(p)$. *If $p_k \to p$ and $p_k \to p'$, then $p = p'$.* For suppose $p \neq p'$, (1.3) holds, and similar relations hold with i, p, x^k, x replaced by j, p', x'^k, x'. (We may evidently take $s' = s$.) Then $h_{ij}(x^k) = x'^k$. x is not in Q_{ij}; for if it were, then $h_{ij}(x) = x'$ as h_{ij} is continuous, and

$$p' = \theta_j(x') = \theta_j(h_{ij}(x)) = \theta_i(x) = p.$$

Similarly x' is not in Q_{ji}. But this contradicts (γ).

These two facts show that the obvious criteria for p_k not $\to p$ hold: If p is in U_i and there are no $\{x^k\}$, x such that (1.3) hold, then p_k does not $\to p$; if $\theta_i(x^k) = p_k$, $x^k \to x$ in Q, and $\theta_i(x) \neq p$, then p_k does not $\to p$. We can now define *open, closed, compact* sets, etc. as usual. A manifold M is closed or open according as the set of points M is compact or not.

Note that *any manifold of class C^r is of class C^s for $s < r$; any open subset of a manifold is a manifold* (with suitably chosen maps); *a finite or denumerable number of manifolds together form a manifold. E_n is a manifold with the obvious maps.*

Given two sets of maps in M, we say they define *C^r-equivalent manifolds* if not only each set separately, but also the two sets together, satisfy the above condi-

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[9] (β) and (γ) correspond to (C^1) and (C^2) in Veblen and Whitehead, loc. cit.

tions. We may speak of a point being the identical point on both manifolds, of a function being identical on them, etc. Two manifolds are *C^r-homeomorphic* if there is a (1-1) correspondence between their points such that, on identifying corresponding points, the two sets of maps define C^r-equivalent manifolds. In other words there is a (proper) (1-1) regular C^r-map of either one into the other (see §2). It is often convenient not to distinguish between two C^r-equivalent manifolds. We may then say for instance "choose maps defining M such that $\cdots$." The number r should then be definitely associated with the manifold; for a C^r-manifold may be C^r-equivalent to a C^s-manifold with $s > r$ (see Theorem 1).

2. **Functions defined in manifolds.** We shall use the words "function" and "map" interchangeably. Let R be an open set in E_m, and let f be a real-valued function defined in R. It is *of class C^r* if it has continuous partial derivatives through the r^{th} order. If R is any subset of E_m,[10] we say f is *of class C^r in the subset R of E_m* if its definition can be extended through an open set containing R so that it is of class C^r there (see AE). It is sufficient that the extension be possible separately about each point of R. If the values of f are points of E_n, we say it is of class C^r (or of class C^r in a subset) if each of its coördinates is.

Suppose f is a function defined in a subset R of a C^r-manifold M, with values in a C^s-manifold N. Let $\theta_i, \chi_j; U_i, V_j; Q_m, Q_n$ be the maps etc. defining M and N respectively.[11] Take any p_0 in R, and say p_0 is in U_i, $q_0 = f(p_0)$ is in V_j. The function $f_i(x) = f(\theta_i(x))$ is defined in the set $R_i = \theta_i^{-1}(R \cdot U_i)$. Suppose that, for some neighborhood U of $x^0 = \theta_i^{-1}(p_0)$ in Q_m, x in $R_i \cdot U$ implies $f_i(x)$ in V_j; set $f_{ij}(x) = \chi_j^{-1}(f_i(x))$, and suppose that f_{ij}, defined in $R_i \cdot U$ and with values in Q_n, is of class C^t $(t \leqq r, s)$. If this is true of each p_0 and each corresponding i, j, we say f is *of class C^t in R*, or in the subset R of M, if R is not open. (If the condition is satisfied at p_0 for one pair (i, j), it is satisfied at p_0 for each such pair (i, j), on account of (δ).) If M and N are analytic, f may be analytic.

Suppose M is of class C^r, $r \geqq 1$. Let x be a point of Q, and let $C_1, \cdots, C_\mu$ be differentiable curves ending at x, whose tangents at x form a set of independent vectors. If x is in Q_{ij}, then parts of these curves, and the vectors, transform under h_{ij} into other such curves and vectors in Q; by (δ), the new vectors are independent. The corresponding curves in M we shall say define a set of *independent directions* at $\theta_i(x)$. If f is a C^1-map of M into N, these curves, and hence "directions," go into curves and directions in N. We define: f is a *regular*[12] map of M into N if it is of class C^1, and any set of independent directions at a point in M goes into such a set in N. If f is defined in a subset of M, we say it is regular if its definition can be extended through an open subset of M so that it is regular there.

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[10] This case does not occur in the fundamental theorems. In this connection, see also H. Whitney, *Differentiable functions* $\cdots$, Trans. Am. Math. Soc., vol. 40 (1936).

[11] We shall always use these symbols in this manner.

[12] In Veblen and Whitehead, loc. cit., it is also required that the map be (1-1).

The map f of M into N is *completely regular* if it is regular, and has the following property: at most two points of M go into any single point of N; if $f(p_1) = f(p_2)$, $p_1 \neq p_2$, then a set of m independent directions at p_1 together with such a set at p_2 go into a set of $2m$ independent directions at $f(p_1)$ in N. This is of course only possible (if f is not (1-1)) if $n \geqq 2m$.

Given the map f of M into N, we define the *limit set* Lf as follows: A point q of N is in Lf if there exist sequences $\{p_k\}$ in M and $\{q_k\}$ in N such that $q_k \to q$, $f(p_k) = q_k$, and the sequence $\{p_k\}$ has no limiting point in M. The map f is *proper* if $f(M)$ does not intersect its limit set: $f(M) \cdot Lf = 0$. If M and N are C^r-m- and C^r-n-manifolds, and f is a (1-1) regular proper C^r-map of M into N, we shall say f *C^r-imbeds* M in N. $f(M)$ is then a C^r-m-manifold in N (see §3).

3. **Manifolds in manifolds.** If f is a regular C^r-map of M into N, we shall call the combination (M, f) a *local C^r-manifold* in N. Each point p of M is then in a neighborhood U in which f is (1-1). In general, the nature of M is determined by the nature of the point set $f(M)$; but this is not necessarily the case. We shall commonly speak of $f(M)$ as a local manifold in N, keeping in mind that M and f must both be given. (But see below.) The *limit set* $Lf(M)$ of $f(M)$ is the limit set Lf. The local manifold is *proper* if f is; has at most *regular singularities* if f is completely regular; is *non-singular* if f is (1-1). (M, f), or the resulting point set $f(M)$, is a *C^r-manifold in N* if it is non-singular and proper. If $f(M)$ is a local manifold in N, and f is of class C^r, we shall say M is *C^r-homeomorphic* with $f(M)$.

We shall show now that if (M, f) is a C^r-manifold in N, then, using the point set $f(M)$ alone, we may determine a C^r-manifold M', which is necessarily C^r-homeomorphic with M. This justifies calling the point set $f(M)$ a manifold in N. Also, setting $N = E_n$, we justify our original definition of a manifold. (See also Theorem 1.) Moreover, as M and $f(M)$ are C^r-homeomorphic, there is in general no harm in identifying them. This justifies the phrase "the manifold M in N."

LEMMA 1. *Let (M, f) be a C^r-m-manifold in the C^r-n-manifold N. Then the subset $f(M)$ of N has the following property. Any q_0 in $f(M)$ is in a neighborhood U in $f(M)$*[13] *such that U is in some V_j, and the points $\chi_j^{-1}(U)$ are given by*

$$y_{m+k} = \psi_k(y_1, \cdots, y_m) \qquad (k = 1, \cdots, n - m) \tag{3.1}$$

(if $y_1, \cdots, y_m$ are suitably chosen rectangular coördinates in E_n), where $(y_1, \cdots, y_m)$ runs through an open set in the $(y_1, \cdots, y_m)$-plane. Moreover, if M' is any subset of N with the above property, then maps in M' determined by (3.1) *make M' a C^r-m-manifold; if, further, $M' = f(M)$, then M is C^r-homeomorphic with M'.*[14]

Given M, f and q_0, say $q_0 = f(p_0)$; we may take a neighborhood U^* of p_0 in M such that U^* is in some U_i and $U = f(U^*)$ is in some V_j (as f is continuous).

[13] That is, q_0 is in U which is in $f(M)$, and U is open in $f(M)$, i.e., no q in U is the limit (in N) of a sequence of points of $f(M) - U$.

[14] It is easily seen that the lemma holds if f is completely regular and proper.

Let $(x_1, \cdots, x_m)$ be rectangular coördinates in E_m. $f_{ij} = \chi_j^{-1}f\theta_i$ maps the open subset $\theta_i^{-1}(U^*)$ of Q_m into Q_n. As f is regular, the matrix $\| \partial y_k/\partial x_l \|$ of partial derivatives of f_{ij} is of rank m at $\theta_i^{-1}(p_0)$; we may suppose that the first determinant is $\neq 0$. Then, taking U^* small enough, (3.1) holds. To show that U is a neighborhood of q_0 in $f(M)$, take any q in U, and suppose there is a sequence $\{q_k\}$ in $f(M) - U$, $q_k \to q$. Say $q = f(p)$, $q_k = f(p_k)$. Suppose there were a subsequence $\{p_{\nu_k}\}$ of $\{p_k\}$ such that $p_{\nu_k} \to p'$ in M. Then $f(p_{\nu_k}) = q_{\nu_k} \to f(p')$, hence $f(p') = q$, and $p' = p$, as f is (1-1). As $p_{\nu_k} \to p$ in U^*, there is an s such that $p_{\nu_{s+k}}$ is in U^* (see §1). But then $q_{\nu_{s+k}}$ is in U, a contradiction. Therefore $\{p_k\}$ has no limit in M. But then f is not proper, again a contradiction.

Next suppose that M' is a subset of N defined by equations (3.1). In each such equation, the domain of definition $R = (y_1, \cdots, y_m)$ is open; we may cover R by spheres, and map Q_m into each sphere and hence into M', defining maps in M'. Using the fact that the h_{ij} in N are of class C^r with non-vanishing Jacobian, it is easily shown that the same is true for the maps in M'. Suppose further that $M' = f(M)$. The equations $y = f_{ij}(x)$ previously considered, when solved as before, give a further set of equations (3.1) and thus another set of maps defining M'; but this set is clearly C^r-equivalent to the other, and thus defines a C^r-homeomorphism between M and M'.

LEMMA 2. *Let M be a C^r-m-manifold in N, and N, a C^r-n-manifold in the C^r-n'-manifold N'. Then M is a C^r-m-manifold in N'. We may replace "manifold" by "local manifold."*

Say f maps M into N and g maps N into N'; then $f' = gf$ maps M into N'. As f and g are regular and of class C^r, so is f'. If f and g are (1-1), so is f'; the same is easily seen to be true with "(1-1)" replaced by "proper."

4. **Functions defined in submanifolds of a manifold.** Let M be a submanifold of N, and let f be defined in one of the manifolds; we propose to study the relation of f to the other manifold. The values of f may be points of another manifold. The results of this section will be used only occasionally.

LEMMA 3. *Let M be a local C^r-m-manifold in the C^r-n-manifold N, let R be an open subset of N, and let f be of class C^r in R. Then f is of class C^r in the subset $R \cdot M$ of M.*

By this we mean, if g is the map of M into N, that $f' = fg$ is of class C^r in that open subset R' of M for which $g(R')$ is in R. Take q_0 in $R \cdot M$, and say $q_0 = g(p_0)$, p_0 in U_i, q_0 in V_j. The hypothesis is that $f^*(y) = f\chi_j(y)$ is of class C^r in a neighborhood of $y^0 = \chi_j^{-1}(q_0)$ in Q_n. As θ_i, g, and χ_j^{-1} are of class C^r, so is

$$f'(\theta_i(x)) = f^*\chi_j^{-1}g\theta_i(x)$$

in a neighborhood of $x^0 = \theta_i^{-1}(p_0)$, as required.

A converse of this lemma is

LEMMA 4. *Let M be a C^r-m-manifold in the C^r-n-manifold N, let R' be an open subset of M, and let f be of class C^r in R'. Then its definition may be*

extended throughout an open subset R of N containing R' so that it is of class C^r there.

As R' is a C^r-m-manifold in N, we may suppose without loss of generality that $R' = M$. By Theorem 1, we may C^r-imbed N in E_ν ($\nu = 2n + 1$); then, by Lemma 2, M becomes a C^r-m-manifold in E_ν. Define f in $R(M)$ by setting $f(p) = f(H(p))$ (Lemma 23). f is now of class C^r in $R(M)$. For, let U be a neighborhood of a point p_0 in M, and let S be the product of $\theta_i^{-1}(U)$ and $E_{\nu-m}$: $z = (x, y)$, x in $\theta_i^{-1}(U)$, y in $E_{\nu-m}$, z in S. S may be considered as a subset of E_n. Setting $f'(z) = f(\theta_i(x))$, f' is obviously of class C^r in S. Let ϕ be a congruent map of $E_{\nu-m}$ into $P(p_0)$, and for any p in U, let $T_p = T_{p,P(p)}$ be the map of $P(p_0)$ into $P(p)$ of §19. Set

$$q = \psi(x, y) = T_{\theta_i(x)}\phi(y).$$

This is a C^r-map of S into E_ν. If $S' = (x, y)$ for $\| y \| <$ some α, and $\phi(O) = p_0$, ψ maps S' into part of $R(M)$. Moreover,

$$x = \theta_i^{-1}(H(q)), \qquad y = \phi^{-1}T_{H(q)}^{-1}\, q \text{ in } \psi(S');$$

hence ψ has an inverse of class C^r in $\psi(S')$. But in $\psi(S')$, by the definitions of f and f', $f(q) = f'(\psi^{-1}(q))$, which shows that f is of class C^r in $R(M)$. Set $R = R(M) \cdot N$; by Lemma 3, f is of class C^r in R.

We remark that the lemmas hold if we replace everywhere "of class C^r" by "analytic." The lemmas show that if M and N are as given, and f is defined in M, then *"f is of class C^r in M" is the same as "f is of class C^r in the subset M of N."*

5. **Admissible sets of maps in a manifold.** Let M be a C^r-m-manifold with maps θ_i. If (a) each θ_i is of class C^r in $\bar{Q}$,[14a] and (b) any compact subset of M has points in common with but a finite number of the U_i, we say the maps form an *admissible set*. If, further, Q' is the sphere concentric with Q and of half the radius, $U_i' = \theta_i(Q')$, and the U_i' cover M, we say the maps form a *completely admissible set*. Any manifold may be defined by a completely admissible set of arbitrarily small maps:

LEMMA 5. *Let M be a C^r-m-manifold, and let $R_1, R_2, \cdots$ be a set of open sets covering M. Then M is C^r-equivalent to a manifold with completely admissible maps θ_i such that each $\bar{U}_i$ is in some R_j.*

Let θ_i^* be the given maps in M. Let $Q_1, Q_2, \cdots$ be the spheres of rational center and rational radius such that each $\bar{Q}_k$ is in Q. Q may be mapped into Q_k by a linear transformation ϕ_k. Set $\theta_{ki}(x) = \theta_i^*(\phi_k(x))$; then the θ_{ki} are defined in $\bar{Q}$, and obviously define a manifold C^r-equivalent to M. Call these maps χ_i, and the corresponding neighborhoods, V_i.

Set $W_i' = V_1 + \cdots + V_i$; then $\bar{W}_i'$ is compact and closed. Each point of $\bar{W}_i'$ is in some W_j', and hence $\bar{W}_i'$ is in the sum of a finite number of the W_j' and is thus in some W_k'. Hence we may pick out a finite or infinite subsequence $W_1, W_2, \cdots$ of the W_i' such that $\bar{W}_i$ is in W_{i+1} and the W_i cover M. Each

[14a] $\bar{Q} = Q$ plus limit points.

point of $\overline{W_i - W_{i-1}}$ is in a $V'_j = \chi_j(Q')$ such that $\bar{V}_j$ lies in some R_k and has no points in W_{i-2}; a finite number of these V'_j may be chosen such that they cover $\overline{W_i - W_{i-1}}$. Choose such neighborhoods for each i; the corresponding maps, arranged in a sequence $\theta_1, \theta_2, \cdots$, obviously have the required properties.

6. **Approximations to functions defined in manifolds.** Let M and N be C^r-m- and C^r-n-manifolds with admissible maps θ_i, χ_j, and let $f(p)$ be a C^r-map of M into N. Take any p_0 in M, and say p_0 is in U_i, $f(p_0)$ is in V_j. Then $f_{ij}(x) = \chi_j^{-1} f \theta_i(x)$ is a C^r-map of part of Q_m into Q_n, with derivatives

$$D_k f_{ij}(x) = \frac{\partial^{k_1 + \cdots + k_m}}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} f_{ij}(x) \qquad (\sigma_k \leqq r),$$

where $\sigma_k = k_1 + \cdots + k_m$. Each $D_k f_{ij}(x)$ is a vector function defined in part of Q_m. Let $\eta(p)$ be a positive continuous function in M, and let $F(p)$ be another C^r-map of M into N. We shall say *F approximates to f in M through the s^{th} order with an error $< \eta$*, or, *F approximates (f, M, s, η)*, if the following is true: For any point p in any U_i there is a j such that $f(p)$ and $F(p)$ are in V_j and

$$\| D_k F_{ij}(p) - D_k f_{ij}(p) \| < \eta(p) \qquad (\sigma_k \leqq s).$$

If $M = E_m$ and $N = E_n$, this reduces to the ordinary definition. For $s = 0$, this inequality is independent of the maps defining M.

Lemma 6. *Given two sets of admissible maps θ_i, θ^*_i in M and two sets χ_j, χ^*_j in N, and given $f(p)$ and $\eta(p)$ as above, there is a positive continuous function $\zeta(p)$ in M such that if F approximates (f, M, s, ζ) in terms of the θ^*_i and χ^*_j, then F approximates (f, M, s, η) in terms of the θ_i and χ_j.*

Let $f_{ij}, f^*_{ij}, F_{ij}, F^*_{ij}$ be the corresponding maps of Q_m into Q_n, and set $u_{ik}(x) = \theta_k^{*-1}(\theta_i(x))$, $v_{jl}(y) = \chi_l^{*-1}(\chi_j(y))$ where defined. Now given i, j, there are numbers k, l such that near a given point

$$f^*_{kl} = \chi_l^{*-1} f \theta^*_k = v_{jl} \chi_j^{-1} f \theta_i u_{i\,k}^{-1} = v_{jl} f_{ij} u_{i\,k}^{-1};$$

hence the derivatives of f^*_{kl} of order $\leqq s$ are polynomials in those of v_{jl}, f_{ij}, and $u_{i\,k}^{-1}$, of order $\leqq s$. It is sufficient to show that all such derivatives are bounded in the neighborhood of any point p_0 of M. p_0 is in a finite number of U_i and U^*_k, and $f(p_0)$ is in a finite number of V_j and V^*_l. Each θ_i etc. is defined in a bounded closed set, and hence its derivatives are bounded; therefore the same is true of the derivatives of the above functions.

We remark that *if f is regular, completely regular, proper, or* (1-1) *regular and proper, then the same is true of any function which approximates to f through the first order closely enough.*

II. The Theorems

We collect here the principal results of the paper (apart from Lemma 23). Using the first two theorems and the results of §9, the remaining theorems may be proved with little difficulty; hence we give these proofs practically in full in this part.

7. **Two types of properties of maps.** The purpose of this section is to explain (e) and (f) of Theorem 2. See also §9. Let M and N be C^r-m- and C^r-n-manifolds, and let f be a fixed C^r-map of M into N ($r \geqq 0$). Let $\eta(p)$ be a positive continuous function in M. We shall say a property Ω of maps of M into N is an *(f, r, η)-property* if the following is true:

(a) Ω is defined for all maps f' which approximate (f, M, r, η).

(b) There is a compact open subset W' of M such that whether any f' has the property Ω depends only on the values of f' in $\overline{W}'$.

(c) If f' approximates (f, M, r, η) and has the property Ω, then for some continuous $\eta'(p)$, $0 < \eta'(p) < \eta(p)$, any map f'' which approximates (f', M, r, η') has the property Ω.

(d) If f' approximates (f, M, r, η), then for an arbitrary continuous $\eta'(p)$, $0 < \eta'(p) < \eta(p)$, there is a map f'' which approximates (f', M, r, η') and has the property Ω.[15]

Before defining the second type of property, we shall consider certain functionals. We suppose now that M and N are analytic manifolds in E_μ and E_ν respectively.

Let $\Delta(p, q)$ be a continuous function of the pair of points p, q of M such that $\Delta(p, p) = 0$ and $\Delta(p, q) > 0$ for $p \neq q$. Let $\zeta(p)$ be a positive continuous function in M. To each C^r-map f of M into E_ν such that

$$\| D_k f(q) - D_k f(p) \| \leqq \Delta(p, q) \qquad (\sigma_k \leqq r), \tag{7.1}$$

(see §6),[16] let there correspond a map $\mathfrak{L}f$ of M into E_ν. We shall say $\mathfrak{L}$ is an *analytic linear $(M, E_\nu, r, \Delta, \zeta)$-functional* if $\mathfrak{L}f$ is analytic, and approximates (f, M, r, ζ), and, if f_1, f_2 and $f_1 + f_2$ satisfy (7.1), then[17]

$$\mathfrak{L}(f_1 + f_2) = \mathfrak{L}f_1 + \mathfrak{L}f_2 .$$

The existence of such a functional is given by Lemma 27. Note that if $\mathfrak{L}$ is an analytic linear $(M, E_\nu, r, \Delta', \zeta')$-functional, then it is an analytic linear $(M, E_\nu, r, \Delta, \zeta)$-functional for $\Delta \leqq \Delta'$, $\zeta \geqq \zeta'$.

We shall say a property Ω is an *$[f, r, \eta, \Delta, \zeta]$-property* if (a), (b) and (c) hold, and also:

(d′) There is a compact open subset W of M containing $\overline{W}'$, and there are C^r-maps $G_1, \cdots, G_h$ of M into E_ν such that $G_i(p) = O$ for p in $M - W$, with the following property. If $\mathfrak{L}$ is any analytic linear $(M, E_\nu, r, \Delta, \zeta)$-functional and if f' approximates (f, M, r, η) and satisfies (7.1), then there is an arbitrarily

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[15] (c) and (d) may be phrased as follows: If $\mathfrak{S}$ is the space of maps of M into N which approximate (f, M, r, η), using a very strong topology, then those maps which have the property Ω form an everywhere dense open subset of $\mathfrak{S}$.

[16] $D_k f(p)$ shall mean some $D_k f_i(x)$, where $p = \theta_i(x)$. If p and q are both in some U_i, we shall use the same i in defining $D_k f(p)$ and $D_k f(q)$.

[17] We add points, etc., by considering them as vectors from the origin O. If $p = \Sigma \alpha_i p_i$ with $\Sigma \alpha_i = 1$, then p is independent of the choice of O.

small $\alpha = (\alpha_1, \cdots, \alpha_h)$ such that

$$f_\alpha(p) = H\mathfrak{L}[f'(p) + \sum \alpha_i G_i(p)]$$

has the property Ω. (We suppose η and ζ are so small that f_α is in $R(N)$ for $|\alpha_i| <$ some $\bar{\alpha} > 0$, and thus H is defined; see Lemma 23.)

Note that any $[f, r, \eta, \Delta, \zeta]$-property is an $[f, r, \eta, \Delta', \zeta']$-property for $\Delta' \geqq \Delta$, $\zeta' \leqq \zeta$; also, for large enough Δ, it is an (f, r, η)-property.

8. **The fundamental theorems.** We state here the two theorems on which most of the other results of the paper are based.

THEOREM 1. *Any C^r-m-manifold ($r \geqq 1$ finite or infinite) is C^r-homeomorphic with an analytic manifold in Euclidean space E_{2m+1}.*

See also Theorem 3 and footnote [32].

THEOREM 2. *Let M and N be analytic m- and n-manifolds in Euclidean spaces E_μ and E_ν respectively. Let f be a C^r-map of M into N ($r \geqq 0$ finite). Let η be a positive continuous function in M. Then there is a C^r-map F of M into N with the following properties:*

(a) *F approximates (f, M, r, η).*

(b) *If $n \geqq 2m$, then F is completely regular.*

(c) *If $n \geqq 2m + 1$, then F is (1-1).*

(d) *F is analytic.*

(e) *Let $\Omega_1, \Omega_2, \cdots$ be (f, r, η)-properties, let $W'_1, W'_2, \cdots$ be the corresponding subsets of M* (§7, (b)), *and suppose any compact subset of M has points in common with at most a finite number of the W'_i. Then F has the properties $\Omega_1, \Omega_2, \cdots$.*

(f) *For some functions $\Delta(p, q)$, $\zeta(p)$ as in* §7, *let $\Omega'_1, \Omega'_2, \cdots$* be $[f, r, \eta, \Delta, \zeta]$-*properties. Then F has these properties.*

In place of (d), (e) *and* (f) *we may have if we choose* (e′): (e) *holds without the finiteness restriction.*

If we are satisfied with a function F of class C^r, we would naturally use (e′) in place of (d), (e) and (f). Note that if f is proper in M, then we can insure that F be proper in M by taking η small enough; then, if $n \geqq 2m + 1$, (b) and (c) show that F is a homeomorphism, and thus $F(M)$ is a C^r- (or analytic) manifold in N. We might generalize the theorem by making $F = f$ at certain isolated points of M, or, if r is replaced by ∞, letting $F(p)$ approximate to $f(p)$ together with higher and higher partial derivatives as p approaches the limit set LM. (Compare AE, Theorem III.) We could also consider manifolds of different classes in different subsets; an example is given in Theorem 5.

9. **Consequences of** (e) **and** (f) **of Theorem 2.** We may give the function F various properties, either because these are of one of the two types, or because they are the logical sums of a denumerable number of such properties. We give some examples below; for the proofs that they are of the required nature, see §§34–35.

(A) and (B) are (b) and (c) of the theorem.

(C) If K is a subset of N which is the sum of at most a denumerable number of sets of zero $(n - m)$-extent (see §17), then $F(M)$ does not intersect K.

(D) If $f(M)$ and N' are local C^1-m- and C^1-n'-manifolds in N, then if $m + n' < n$, the local manifold $F(M)$ does not intersect N', while if $m + n' \geqq n$, $N^* = F(M) \cdot N'$ is a local $(m + n' - n)$-manifold in N. At each point p of N^*, there are n independent directions in N, each being parallel to $F(M)$ or to N'. If $f(M)$ and N' are non-singular, or proper, so is N^*.

10. **A further imbedding theorem; Riemannian manifolds.** We may replace E_{2m+1} by E_{2m} in Theorem 1 as follows:

THEOREM 3. *Any C^r-m-manifold M ($r \geqq 1$ finite or infinite) is C^r-homeomorphic with a proper analytic local manifold with at most regular singularities in E_{2m}.*

To prove this, let M' be a C^r-homeomorphic analytic manifold in E_{2m+1}. Let $\rho(p)$(p in M') be the smaller of (a) 1, (b) the reciprocal of the distance from p to a fixed point of E_{2m+1}, (c) the distance from p to the limit set LM' if $LM' \neq 0$. Let q_0 be a fixed point distinct from the origin O in E_{2m}, and set $f(p) = \rho(p)q_0$ in M'. This is a continuous map of M' into E_{2m} such that $f(p) \neq O$, and either $Lf(M')$ is void or $Lf(M') = O$; hence f is proper. Let F be the analytic completely regular proper map given by Theorem 2 with $N = E_{2m}$; $F(M')$ is the required local manifold.

THEOREM 4. *Any C^r-manifold M ($r \geqq 1$ finite or infinite) may be given an analytic Riemannian metric, the coefficients of the fundamental quadratic form being of class C^r in terms of the original maps defining M.*

This follows from Theorem 1 or 3 on using the ds^2 of E_{2m+1} or E_{2m}.

11. **An extension theorem.** Suppose a closed subset of a manifold M is mapped into a manifold N; can the map be extended over the rest of M so as to be differentiable? Or, M might be replaced by a manifold M with boundary B, and the closed subset, by B. An answer is given by the following theorem.

THEOREM 5. *Let A be a separable metric space, let B be a closed subset of A, and let $M = A - B$ be a C^r-m-manifold ($r \geqq 1$ finite or infinite).*[18] *Let N be a C^s-n-manifold ($s \geqq r$), and let f be a continuous map of B into N. Suppose f can be extended so as to be continuous throughout M.*[19] *Then there is a map F of A into N with the following properties:*

(a) *F is continuous in A and of class C^r in M; $F = f$ in B.*

(b) *If $n \geqq 2m$, F is completely regular in M.*

(c) *If $n \geqq 2m + 1$, F is (1-1) in M.*

Suppose, in addition, that A is a $C^{r'}$-m-manifold,[20] *$1 \leqq r' \leqq r$; f is of class C^t in the subset B of A (see §2), $1 \leqq t \leqq r'$. Then we have also*

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[18] We suppose that continuity in M agrees with continuity in A.

[19] This is always possible if $N = E_n$. See for instance Kuratowski, *Topologie* I, p. 211, or Alexandroff-Hopf, *Topologie* I, p. 76.

[20] We suppose that the maps defining M are a subset of those defining A.

(d) *F is of class C^t in A.*

(e) *If $n \geqq 2m$ and f is regular [completely regular] in B, then F is regular [completely regular] in A.*

(f) *If $n \geqq 2m + 1$ and f is regular and* (1-1) *in B, then F is regular and* (1-1) *in A.*

We might also apply (e) and (f) of Theorem 2. If the extension of f over M is proper in M [in A], we can make F proper in M [in A]. If N is analytic, we may make $f(M)$ analytic, etc., as in Theorem 2.

By Theorem 1, there are analytic manifolds M' and N' in E_μ and E_ν, C^r- and C^s-homeomorphic with M and N respectively. f[20a] gives a map f' of M' into N'. Choose $\eta(p)$ positive and continuous in M so that $\eta(p) \to 0$ as $p \to B$. Applying Theorem 2 with its r replaced by 0, we replace the extension f' over M' by a function F'; the resulting map F of M into N is of class C^r and has the properties (a), (b) and (c) (setting $F = f$ in B).

Now suppose that A is a $C^{r'}$-manifold, $1 \leqq r' \leqq r$. Let A_1' be a $C^{r'}$-homeomorphic analytic manifold in E_μ; then M is $C^{r'}$-homeomorphic with the corresponding subset M_1' of A_1'. Let M'' be an analytic manifold in E_μ', C^r-homeomorphic with M. The map g of M'' into M_1' thus defined is of class $C^{r'}$. We may choose $\eta(p)$ positive and continuous in M so that $\eta(p) \to 0$ as $p \to B$, and so that, considering $\eta(p)$ in M'', if g'' approximates (g, M'', r', η), then g'' is a homeomorphism and g'' (M'') does not intersect $B' = A_1' - M_1'$. Let g'' be such a function which is analytic (Theorem 2); the resulting map g' of M into $g'(M)$ is a C^r-homeomorphism. Letting g', in B, be the map already given, g' is (1-1) in A. From Lemma 10 below it is seen that g' is of class $C^{r'}$ in A. It is regular, and taking $\eta(p)$ small enough insures that it is proper; hence *g' is a $C^{r'}$-homeomorphism in A and a C^r-homeomorphism in M.* Let $A' = g'(A)$, $M' = g'(M)$. M' (but not A') is analytic. Let N' and f' be as before.

f' is continuous in A'; considering the values of f' in B' alone, it is of class C^t in the subset B' of E_μ (Lemma 4). Suppose that we have proved Lemma 7; then there is a function F'' of class C^t in $R(A')$ (see Lemma 23) which equals f' together with derivatives of order $\leqq t$ in B', and such that $F''(M')$ is in $R(N')$.[21] Then $F''' = HF''$ (see Lemma 23) is of class C^t in A', and maps A' into N'. Define $\eta(p)$ in M' as before, and let F' be the approximation to F''' in M' given by Theorem 2. Set $F' = F''$ in B'; then F' is of class C^t in A', as is easily seen from Lemma 10. The resulting map F of A into N is of class C^t in A and of class C^r in M, and has the properties (a) through (d). The regularity condition in (e) is satisfied automatically; we obtain complete regularity if f is completely regular in B by applying the method of proof of (A) of §9 in using Theorem 2. Suppose $n \geqq 2m + 1$ and f is (1-1) in B. Then $F(B)$ is the sum of a denumerable number of sets of zero $(m + 1)$-extent, by Lemmas 13, 14 and 15, and hence we may make $F(M)$ avoid $F(B)$ by applying (C) of §9.

There remains to prove

[20a] I.e. the extension of f over M.

[21] If $N = E_\nu$, we could obviously avoid Lemma 7.

LEMMA 7. *Let A^* be a subset of the open set R in E_μ, $\bar{A}^* \cdot R = A^* \cdot R$, and let B^* be a subset of A^*, $\bar{B}^* \cdot A^* = B^* \cdot A^*$. Let $\zeta(p)$ be a positive continuous function in A^*. Let f' be a continuous map of A^* into E_ν, and let f', considered in B^* alone, be of class C^t in B^*. Then there is a C^t-map F of A^* into E_ν which equals f' in B^*, together with partial derivatives of order $\leqq t$, and approximates $(f', A^* - B^*, 0, \zeta)$.*

We may suppose that $E_\nu = E_1$, the space of real numbers. By a direct application of the method of proof of AE Lemma 3,[22] we find a function f which is continuous in R, is of class C^t in a neighborhood R' of B^* (R' in R), $= f'$ in B^*, and approximates $(f', A^* - B^*, 0, \zeta)$. Let R'' be a neighborhood of B^* such that $S = \bar{R}'' \cdot A^*$ is in R'. Set

$$A_s = E_\mu - R \qquad (-1 \leqq s \leqq t - 1),$$

$$A = A' = A_s = A_0 + B^* \qquad (s \geqq t),$$

$$B_s = A^* - S \qquad (0 \leqq s \leqq t - 1),$$

$$B = B_s = A^* - B^* \qquad (s \geqq t).$$

The conditions of AE Theorem III are seen to be satisfied; the function F given by the theorem (if $\epsilon(x)$ is small enough) has the required properties.

12. A deformation theorem.

We first introduce some definitions. Let M and N be C^r-m- and C^r-n-manifolds, let I be the closed interval $(0, 1)$, and let I' be an open interval containing I. Let $M' = M \times I'$ be the product of M and I'; this is a C^r-$(m + 1)$-manifold with easily defined maps. Let f be a regular C^r-map of M' into N. For each p in M and t in I', set $\phi_t(p) = f(p \times t)$. Each ϕ_t is a regular C^r-map of M into N. Set $M_0 = \phi_0(M)$, $M_1 = \phi_1(M)$; these are local C^r-manifolds in N. If, given M_0 and M_1, there exist such M, I' and f, we shall say the set of maps ϕ_t, t in I, forms a *regular C^r-deformation* of M_0 into M_1. A regular deformation such that M' has at most regular singularities, we shall call *completely regular*; one in which each ϕ_t is (1-1), we call *topological*; one in which f is (1-1), we call *completely topological*. If there is such a map f which is merely continuous, it defines a *deformation* of M_0 into M_1.

THEOREM 6. *Let M_0 and M_1 be C^r-homeomorphic local C^r-m-manifolds in the C^r-n-manifold N ($r \geqq 1$ finite or infinite, $n \geqq 2m + 2$). Suppose there exists a*

[22] Let f_1 be a continuous extension of f' throughout R. Take a subdivision of $R - B^*$ as in AE §8, and define an extension f_2 of class C^t of f', considered in B^* alone, throughout R. (In using AE §9, we take for x^ν a point of B^* whose distance from y^ν is less than twice the distance from y^ν to B^*). Define the ϕ_k in $R - B^*$ as in AE. Given a neighborhood R^* of B^*, R^* in R, let ϕ_{i_k} be those functions which are $\neq 0$ somewhere in $R - R^*$, and set

$$f = f_2 + \sum \phi_{i_k}(f_1 - f_2).$$

As $\sum \phi_{i_k} = 1$ in $R - R^*$, $f = f_1$ in $R - R^*$; also $f = f_2$ in a neighborhood of B^*. For R^* small enough, f evidently approximates to f' as required.

deformation of M_0 into M_1 in N.[23] *Then there is a regular C^r-deformation of M_0 into M_1. If M_0 and M_1 are completely regular, so is the deformation. If M_0 and M_1 are non-singular, it is topological. If, further, $n \geqq 2m + 3$ and M_0 and M_1 do not intersect, it is completely topological.*[24]

We shall not consider the question of proper maps, but merely note that any map of a closed manifold is proper. The following lemma is necessary.

LEMMA 8. *Let M be a local C^r-m-manifold in the C^r-n-manifold N, $n \geqq 2m + 1$. Then there is a vector function $v(p)$ of class C^r in M such that for each p, $v(p)$ is independent of the directions in M at p.*[25]

The lemma may be proved most simply by imbedding N in E_ν (Lemma 19), triangulating M, and defining $v(p)$ successively over the 0-cells, 1-cells, $\cdots$ of M. It may also be proved easily by the methods in this paper.

Let $v_i(p)$ be a vector function in M_i as in the lemma, $i = 0, 1$. Let us C^r-imbed N in some E_ν (Lemma 19); then $v_0(p)$ becomes a vector $v_0'(p)$ in E_ν parallel to N but independent of M at p. Set

$$g_0(p \times t) = p + tv_0'(p);$$

this gives a C^r-map of $M' = M \times I'$ into E_ν. For $t <$ some $\zeta(p)$, $g_0(p \times t)$ lies in $R(N)$ (see Lemma 23); set $g_0'(p \times t) = Hg_0(p \times t)$. Let $f = g_0'$ in M_0; g_0' also defines the derivatives of f of order $\leqq r$ in the subset M_0 of M'. By the choice of v_0, f is a regular map of the subset M_0 of M' into N (see §2). Define f similarly in M_1.

By hypothesis, there is a continuous map f' of M' into N which agrees with f in M_0 and in M_1. We now apply Theorem 5 with A, B, t replaced by M', $M_0 + M_1$, r. This gives a regular C^r-map of M' into N, and hence a regular C^r-deformation of M_0 into M_1. The other statements in the theorem follow at once from Theorem 5, except for the statement on topological deformations; we leave the proof of this to §36.

13. **Spheres bounding differentiable cells.** Let S be the unit m-sphere in E_{m+1}, let Q' be the interior of S, and let Q be the interior of a larger concentric m-sphere. Let f be a regular C^r-map of S into N; we shall call $S_1 = f(S)$ *a local C^r-m-sphere* in N; we leave out "local" if f is (1-1). If f may be extended throughout Q so as to be regular and of class C^r, we say S_1 *C^r-bounds regularly* the $(m + 1)$-cell $f(Q')$. If f is completely regular, or (1-1), we call the bounding *completely regular*, or *topological*. If there exists an f which is merely continuous, we say S *bounds* a cell in N.

THEOREM 7. *Any local C^r-m-sphere S in the C^r-n-manifold N ($r \geqq 1$, $n \geqq 2m + 2$) which bounds a cell in N, C^r-bounds regularly a cell in N. If S has at*

[23] This is of course always the case if $N = E_n$.

[24] If N is an analytic manifold in E_ν, we may make the map of M' analytic except over M_0 and M_1 (see Theorem 2). We shall strengthen the theorem in §36.

[25] The lemma holds if $N = E_n$ and $n \geqq 2m$, as will be shown in a paper on "sphere-spaces." "Vector function" means here "direction function."

most regular singularities, the bounding is completely regular; if $n \geqq 2m + 3$ and S is non-singular, the bounding is topological.[26]

The proof runs almost exactly like that of Theorem 6.

14. **Examples** (A) Let M be the interval $(-\infty, +\infty)$. Let $a_0, a_1, \cdots$ be a sequence of points dense in E_n, $n \geqq 2$. For each integer $i \geqq 0$, set $f(i) = a_i$, and let f map the interval $(i, i + 1)$ into the segment $\overline{a_i a_{i+1}}$. Set $f(-t) = f(t)$. This is a continuous map of M into a subset of E_n. Set $\eta(t) = 1/(1 + |t|)$. Theorem 2 then gives, if $n = 2$, an analytic curve everywhere dense in the plane, and if $n \geqq 3$, a non-singular analytic curve dense in E_n. On applying (C) of §9, we may (say in E_3) make the curve avoid all rational lines; we may also replace the curve by a denumerable number of such curves which are non-intersecting.

(B) Let B be the unit circle in the plane, with interior M, and set $A = B + M$. Map B into the whole of the sphere S_n of dimension $n \geqq 4$. Any continuous map of B into S_n may be extended continuously over M; hence, by Theorem 2, there is an analytic regular map F of M into S taking on the given boundary values; if $n \geqq 5$, $F(M)$ is non-singular. As in (A), we may make $F(M)$ avoid sets of $(n - m)$-extent zero, may find a denumerable number of non-intersecting surfaces of this sort if $n \geqq 5$, etc.

III. The Imbedding Theorem

In this part we shall prove Lemma 19; this is Theorem 1, except for the analyticity condition. The proof of Theorem 1 will be completed in Part V. The present proof falls into two parts. We first show, in Lemma 12, practically, that Theorem 2 with the conclusions (a) and (e′) alone holds; we then show that (B) and part of (A), §9, hold. This, with the lemma, gives the imbedding theorem. We first give some lemmas of a general nature.

15. **Some general lemmas.** The lemmas which follow are mostly simple extensions of results from AE.

Lemma 9.[27] *Let $f(p)$ be a C^r-map $(r \geqq 0)$ of the open set R in E_m into E_n, and let $\eta(p)$ be positive and continuous in R. Then there is an analytic map $F(p)$ in R which approximates (f, R, r, η).*

If $n = 1$, this follows at once from AE Lemma 6. We define open sets $R_1, R_2, \cdots$ as in that lemma, and let ϵ_i be the lower bound of $\eta(p)$ for p in R_{i+1}. For the general case, we apply the lemma separately for each coördinate in E_n.

Lemma 10. *Let A be a closed subset of the C^r-manifold M, let $\eta(p)$ be positive and continuous in $M - A$, let $\eta(p) \to 0$ as p approaches any point of A, and let*

[26] Compare footnote[24].

[27] See also Lemmas 22 and 26. We may make $F(p)$ approximate to $f(p)$ to higher and higher orders as p approaches the boundary of R; see AE Lemma 6.

$f(p)$ be a C^r-map of M into the C^r-manifold N. If F approximates $(f, M - A, r, \eta)$ and $F = f$ in A, then F is of class C^r in M.

Suppose first $M = E_m$, $N = E_1$; the theorem then follows from AE Lemma 1 (see AE, end of §11). The general case is an immediate consequence of this.

LEMMA 11.[28] *Let A be a subset of the open set R in E_m, with $\bar{A}\cdot R = A\cdot R$, and let $f(p)$ be of class C^r $(r \geqq 0)$ in the subset A of R. Then the definition of f may be extended throughout R so it is of class C^r there. The values of $f(p)$ may be points of E_n.*

If $R = E_m$ and f is real-valued, the proof is given by AE Lemma 2. If $R \neq E_m$, the proof needs only a slight alteration; or we may use AE Theorem III. If the values of f are points of E_n, we apply the lemma to each coördinate separately.

16. **Maps of a manifold with given properties.** The first of the three lemmas giving the imbedding theorem is the following.

LEMMA 12. *Let f be a C^r-map of the C^r-m-manifold M into the C^r-n-manifold N, let $\eta(p)$ be positive and continuous in M, and let $\Omega_1, \Omega_2, \cdots$ be (f, r, η)-properties. Then there is a C^r-map F of M into N which approximates (f, M, r, η) and has these properties.*

It is clear that there is a sequence of positive continuous functions $\eta_i''(p)$ such that if $f_0 = f$ and f_i approximates $(f_{i-1}, M, r, \eta_i'')$, then $F = \lim f_i$ exists and approximates (f, M, r, η). We shall choose functions $\eta_{1j}(j \geqq 1)$, f_1, $\eta_{2j}(j \geqq 2)$, $f_2, \cdots$ in that order so that if $\eta_i(p)$ for each p is the smallest of $\eta_{1i}(p), \cdots, \eta_{ii}(p), \eta_i''(p)$, then f_i approximates (f_{i-1}, M, r, η_i), and $f_j(j \geqq i)$ and F have the property $\Omega_i(i = 1, 2, \cdots)$. Suppose we have found these functions through f_{i-1}. For each p in M, let $\eta_i'(p)$ be the smallest of the numbers $\eta'(p)$ of (c), §7, for the properties $\Omega_1, \cdots, \Omega_{i-1}$; then if f' approximates (f_{i-1}, M, r, η_i'), f' will have the properties $\Omega_1, \cdots, \Omega_{i-1}$. Choose $\eta_{ij}(p)(j \geqq i)$ so that if $f_{i-1}' = f_{i-1}$ and f_j' approximates $(f_{j-1}', M, r, \eta_{ij})(j \geqq i)$, then $F' = \lim f_j'$ approximates (f_{i-1}, M, r, η_i'). Define η_i, and let $f_i(p)$ approximate (f_{i-1}, M, r, η_i) and have the property Ω_i, by (d), §7. We thus find all the above functions, and the function F has the required properties.

17. **The k-extent of a set.** Let A be a subset of E_m. We shall say the *k-extent of A is finite* if there is a number G such that if $0 < \epsilon < 1$, then there are sets $A_1, \cdots, A_\nu$ of diameter $< \epsilon$ which cover A, and $\nu\epsilon^k < G$. If M is a C^1-m-manifold with admissible maps θ_i etc., and A is a subset of M, we say the *k-extent of A is finite* if A is compact (and hence is in a finite number of the U_i), and each $\theta_i^{-1}(A \cdot U_i)$ is of finite k-extent in E_m. The subset A of E_m is of *k-extent zero* if for any $\epsilon' > 0$ there is a $\delta > 0$ such that if $0 < \epsilon < \delta$, then there are sets $A_1, \cdots, A_\nu$ of diameter $< \epsilon$ covering A, and $\nu\epsilon^k < \epsilon'$. Similarly

[28] This lemma is not needed in the proof of the fundamental theorems, though it is useful in Lemma 17. When we have proved the imbedding theorem, we may show easily that the lemma holds with E_m and E_n replaced by a C^r-manifolds (see §4).

if A is a subset of M. A set of zero k-extent ($k \leqq 0$) or of finite l-extent ($l < 0$) is vacuous. These definitions (in M) are independent of the admissible set of maps defining M (see Lemma 15). The sum of a finite number of sets of finite [zero] k-extent is of finite [zero] k-extent.

LEMMA 13. *A bounded subset A of E_k has finite k-extent; if A contains inner points, then it has not zero k-extent.*

This is obvious on cutting up E_k into equal cubes of any desired size. Equally obvious is

LEMMA 14. *If A has finite k-extent, it has zero $(k+1)$-extent.*

LEMMA 15. *If M and N are C^1-m- and C^1-n-manifolds, f is a C^1-map of M into N, and A is a subset of M*[29] *of finite* [*zero*] *k-extent, then $f(A)$ is of finite* [*zero*] *k-extent.*

Consider first the case of finite k-extent. A is in a finite number of the U_i, and $f(A)$ is in a finite number of the V_j; hence we can put $A = A_1 + \cdots + A_t$, A_s in $U_{i(s)}$, $f(A_s)$ in $V_{j(s)}$. Set $B_s = \theta_{i(s)}^{-1}(A_s)$; then B_s is of finite k-extent in E_m. It is sufficient to show that if $g_s = \chi_{j(s)}^{-1} f \theta_{i(s)}$, then $g_s(B_s)$ is of finite k-extent in E_n. g_s is of class C^1 in the compact set B_s; hence, for some number μ, any subset B' of B_s is mapped by g_s into a set $g_s(B')$ whose diameter is at most μ times that of B'. Set $G^* = \mu^k G$. (G corresponds to B_s.) Given ϵ, $0 < \epsilon < 1$, set $\epsilon_1 = \epsilon/\mu^k$, and divide B_s into sets $B_{s1}, \cdots, B_{s\nu}$ of diameter $< \epsilon_1$, so that $\nu\epsilon_1^k < G$. Then $g(B_{s1}), \cdots g(B_{s\nu})$ cover $g(B_s)$, these sets are of diameter $< \epsilon$, and $\nu\epsilon^k < G^*$, as required.

Consider now the case of zero k-extent. Define the B_s etc. as before. Given $\epsilon' > 0$, set $\epsilon_1' = \epsilon'/\mu^k$, and choose $\delta > 0$ so that if $0 < \epsilon_1 < \delta$, then there are sets $B_{s1}, \cdots, B_{s\nu}$ of diameter $< \epsilon_1$ covering B_s, and $\nu\epsilon_1^k < \epsilon_1'$. Now take any ϵ, $0 < \epsilon < \delta$, and set $\epsilon_1 = \epsilon/\mu$. Define the sets B_{si}; then $g(B_{s1}), \cdots, g(B_{s\nu})$ have the required properties.

18. **Transformations of one set away from another.** Let R and R' be open subsets of E_m and E_h, and for each $\alpha = (\alpha_1, \cdots, \alpha_h)$ in R' let T_α be a C^r-map of R into E_n. Let $x' = T_\alpha(x)$ be of class C^1 in terms of the $n + h$ variables (x, α)(x in R). If for each x in R and α in R' the vectors

$$\partial x'/\partial\alpha_1, \cdots, \partial x'/\partial\alpha_h$$

are independent, we shall say the T_α form an *h-parameter family* of C^r-maps. We may also define such families of maps of one manifold into another in an obvious manner; Lemma 16 holds for such families also.

LEMMA 16. *Let $T_\alpha(x)$ be an h-parameter family of C^1-maps of R into E_n, and let A and B be closed sets in R and E_n of finite k-extent and zero $(h-k)$-extent respectively. Then for some α in R', $T_\alpha(A)$ does not intersect B.*[29a]

We may suppose that $0 \leqq k \leqq h$. For some α^0 in R' and $\eta > 0$, all points α

[29] Note that M may be replaced by any open subset R of M which contains A. It is evidently sufficient that the map satisfy a Lipschitz condition.

[29a] We may evidently interchange "finite" and "zero".

within η of α^0 are in R'. Take $\xi \geqq 1$ so that if A^*, in A, is of diameter $< \epsilon$, then $T_{\alpha^0}(A^*)$ is of diameter $< \xi\epsilon$. The condition that the $\partial x'/\partial\alpha_i$ are independent for x in A and $\alpha = \alpha^0$ is equivalent to the condition that for some $\beta > 0$, any directional derivative for x in A and $\alpha = \alpha^0$ is a vector of length $\geqq \beta$. Hence we may take ζ, $0 < \zeta < \eta$, such that for any $\epsilon > 0$, if A^* and B^* are subsets of A and B of diameter $< \epsilon$, α and α' are points of R' within ζ of α^0, and the distance from α to α' is $> 4\xi\epsilon/\beta$, then either $T_\alpha(A^*)\cdot B^* = 0$ or $T_{\alpha'}(A^*)\cdot B^* = 0$.

Let G be the number corresponding to A, and take $\epsilon' \leqq (\beta\zeta)^h/[2^h(4\xi)^hG]$. Find $\delta < 1$ corresponding to B and ϵ'. Choose an ϵ, $0 < \epsilon < \delta$. Then we may set

$$A = A_1 + \cdots + A_\nu, \qquad B = B_1 + \cdots + B_\sigma,$$

so that the A_i cover A, the B_j cover B, the diameter of each A_i and each B_j is $< \epsilon$, and so that

$$\nu\epsilon^k < G, \qquad \sigma\epsilon^{h-k} < \epsilon'.$$

Let Z_{ij} be the set of all points α within ζ of α^0 such that $T_\alpha(A_i)\cdot B_j \neq 0$; then, by the choice of ζ, the diameter of Z_{ij} is $< 4\xi\epsilon/\beta$, and hence its ordinary h-volume is $< (4\xi\epsilon/\beta)^h$. Therefore the ordinary h-volume of all the Z_{ij} is less than

$$\nu\sigma(4\xi\epsilon/\beta)^h = (4\xi)^h(\nu\epsilon^k)(\sigma\epsilon^{h-k})/\beta^h < (4\xi)^hG\epsilon'/\beta^h \leqq \zeta^h/2^h.$$

Hence there is an α within ζ of α^0 which is in no Z_{ij}; for this α, $T_\alpha(A)$ does not intersect B.

19. **The transformation $T_{p,P}$.** Let P_0 be a fixed k-plane through a fixed point p_0 in E_n. Corresponding to any k-plane P not perpendicular to P_0 and any point p of P we may let correspond a non-homogeneous orthogonal transformation of space $T_{p,P}$ which carries p_0 and P_0 into p and P, and is analytic in p and P (see §24), for instance as follows. Let P_0' and P' be the parallel planes through the origin. Let $v_1, \cdots, v_n$ be mutually orthogonal unit vectors such that $v_1, \cdots, v_k$ are in P_0'. The following rules determine Tv_i. The Tv_i are mutually orthogonal unit vectors. If $1 \leqq i \leqq k$ and P_i' is the plane in P' determined by the projections $v_1', \cdots, v_i'$ of $v_1, \cdots, v_i$ into P', then $Tv_1, \cdots, Tv_i$ are in P_i' and determine the same orientation in P_i' as $v_1', \cdots, v_i'$; for each j ($j = k+1, \cdots, n$), if P_j' is the plane determined by P' and $v_{k+1}, \cdots, v_j$, then $Tv_{k+1}, \cdots, Tv_j$ are parallel to this plane, and $Tv_1, \cdots, Tv_j$ determine the same orientation in it as $Tv_1, \cdots, Tv_k, v_{k+1}, \cdots, v_j$. $T_{p,P}$ is the unique linear transformation carrying p_0 into p and v_i into Tv_i.

20. **(1-1) maps and properties.** We can now prove

LEMMA 17. *If f is a regular C^r-map ($r \geqq 1$) of the C^r-m-manifold M into E_n, $n \geqq 2m + 1$, and η is a sufficiently small positive continuous function in M, then the property of maps f' which approximate (f, M, r, η) of being* (1-1) *and avoiding*

a fixed point b_0 is the logical sum of a denumerable number of (f, r, η)-properties.[30]

Let $\theta_1, \theta_2, \cdots$ be a completely admissible set of maps defining M; we choose them so that if $\overline{U}_i$ and $\overline{U}_j$ have common points, then f is (1-1) in $\overline{U}_i + \overline{U}_j$. (This is simple; see §5.) Consider all sets $U_i + U_j$ such that $\overline{U}_i \cdot \overline{U}_j = 0$; we arrange these in a sequence $W_1, W_2, \cdots$. Take $\eta(p)$ so small that any map f' which approximates (f, M, r, η) is regular and (1-1) in all $\overline{U}_i + \overline{U}_j$ for which $\overline{U}_i \cdot \overline{U}_j \neq 0$. Take any k, and say $W_k = U_i + U_j$; let Ω_k hold for the map f' if $f'(\overline{U}_i') \cdot f'(\overline{U}_j') = 0$ and b_0 is not in $f'(\overline{U}_i')$. As the U_i' cover M, the property of f' being (1-1) and avoiding b_0 is the sum of the properties $\Omega_1, \Omega_2, \cdots$.

It remains to show that Ω_k is an (f, r, η)-property. (a) of §7 holds; (b) holds with W' replaced by $W_k' = U_i' + U_j'$. As $f'(\overline{U}_i')$ and $f'(\overline{U}_j') + b_0$ are bounded closed subsets of E_n, (c) is obvious; it remains to prove (d). Let f' approximate (f, M, r, η). Let $\lambda'(x)$ be a function of class C^r in E_m which $= 1$ in $\bar{Q}_m'$ and $= 0$ in $E_m - Q_m$. (Such a function is given by Lemma 11, replacing A by $\bar{Q}_m' + (E_m - Q_m)$; or the function may be constructed directly without great difficulty.) Then $\lambda(p) = \lambda'(\theta_i^{-1}(p))$ in U_i and $= 0$ in $M - U_i$, is of class C^r in M, and $= 1$ in $\overline{U}_i'$. For any vector v in E_n, set

$$f(p, v) = f'(p) + \lambda(p)v.$$

Given an arbitrary $\eta'(p)$, we may choose $\beta > 0$ so that if[31] $\| v \| < \beta$, then $f(p, v)$ approximates (f', M, r, η'). By Lemmas 13, 14 and 15, $f'(\overline{U}_j') + b_0$ is of zero $(m + 1)$-extent and $f'(\overline{U}_i')$ is of finite m-extent. The transformations $T_v q = q + v$ with $\| v \| < \beta$ form an n-parameter family in E_n; by Lemma 16, there is a v_0 such that $T_{v_0} f'(\overline{U}_i')$ does not intersect $f'(\overline{U}_j') + b_0$. $f'' = f(p, v_0)$ with this v_0 is then the required approximation; for $f'' = f' + v_0$ in $\overline{U}_i'$ and $= f'$ in $\overline{U}_j'$.

21. **Regular maps and properties.** The final lemma is

LEMMA 18. *If f is a C^r-map $(r \geqq 1)$ of the C^r-m-manifold M into E_n, $n \geqq 2m$, and η is a positive continuous function in M, then the property of maps f' which approximate (f, M, r, η) of being regular is the logical sum of a denumerable number of (f, r, η)-properties.*

Let $\theta_1, \theta_2, \cdots$ be completely admissible maps defining M, and let Ω_i hold for f' if f' is regular in $\overline{U}_i'$; then if $\Omega_1, \Omega_2, \cdots$ hold, f' is regular in M. Each Ω_i satisfies the conditions (a), (b) and (c) of §7; to show that it is an (f, r, η)-property, we must show that it satisfies (d). Set $g(x) = f'(\theta_i(x))$ in Q_m; this is a C^r-map of Q_m into E_n. As in the last §, it is sufficient to show that for an arbitrary $\zeta > 0$ there is a map $g'(x)$ of Q_m into E_n which approximates (g, Q_m, r, ζ) and is regular in Q_m'. We then set

$$f''(p) = f'(p) + \lambda(p)[g'(\theta_i^{-1}(p)) - f'(p)] \text{ in } U_i,$$

and $f''(p) = f'(p)$ in $M - U_i$; as $f''(p) = g'(\theta_i^{-1}(p))$ in $\overline{U}_i'$, it is regular in $\overline{U}_i'$.

[30] In §34 we shall express the property as a sum of $[f, r, \eta, \Delta, \zeta]$-properties.

[31] $\| v \|$ is the length of v.

If $r = 1$, let $g_0(x)$ be a function of class C^2 which approximates (g, Q_m, r, ζ') (Lemma 9); otherwise, set $g_0(x) = g(x)$. We shall find functions $g_1(x), \cdots, g_m(x) = g'(x)$ such that $g_i(x)$ approximates $(g_{i-1}, Q_m, r, \zeta')$, and so that the vectors $\partial g_i/\partial x_1, \cdots, \partial g_i/\partial x_i$ are independent in $\bar{Q}'_m$; if ζ' is small enough, g' is the required function.

Suppose we have found g_{i-1}. For any vector v in E_n, set

$$g_i(x, v) = g_{i-1}(x) + x_i v;$$

then

$$\frac{\partial g_i(x, v)}{\partial x_i} = \frac{\partial g_{i-1}(x)}{\partial x_i} + v, \qquad \frac{\partial g_i(x, v)}{\partial x_j} = \frac{\partial g_{i-1}(x)}{\partial x_j} \qquad (j \neq i).$$

As the $\partial g_{i-1}(x)/\partial x_j$, $j = 1, \cdots, i-1$, are independent in $\bar{Q}'_m$, we need merely show that there is an arbitrarily small v_0 such that $\partial g_i(x, v_0)/\partial x_i$ is independent of these vectors at each point of $\bar{Q}'_m$; we then set $g_i(x) = g_i(x, v_0)$. By Lemma 13, it is sufficient to show that the vectors v, $\| v \| \leqq 1$, which do not have the required property, form a set of zero n-extent in E_n. Given the point x^0 of $\bar{Q}'_m$, it is sufficient to show that the vectors not having the required property in a closed neighborhood S of x^0 are of zero n-extent; for a finite number of such sets S cover $\bar{Q}'_m$.

Let $P(x)$ be the $(i-1)$-plane through the origin O in E_n determined by $\partial g_{i-1}(x)/\partial x_1, \cdots, \partial g_{i-1}(x)/\partial x_{i-1}$, and set $P_0 = P(x^0)$. (If $i = 1$, $P(x) = O$.) Choose S so that $P(x)$ is not perpendicular to P_0 for x in S. Let $y_1, \cdots, y_n$ be rectangular coördinates (with origin O) in E_n such that P_0 is the $(y_1, \cdots, y_{i-1})$-plane. Let $E = E_{m+i-1}$ be the space with coördinates $(x_1, \cdots, x_m, y_1, \cdots, y_{i-1})$. Set

$$v(x) = \frac{\partial g_{i-1}(x)}{\partial x_i}, \qquad K = \max \| v(x) \| \ (x \text{ in } S).$$

Let D be the subset of E with x in S, $\| \bar{y} \| \leqq K + 1$, where $\bar{y} = (y_1, \cdots, y_{i-1})$. Let $T_x = T_{O, P(x)}$ be the transformation of §19 leaving O fixed and carrying P_0 into $P(x)$. Given $\bar{y}$, set $y = (y_1, \cdots, y_{i-1}, 0, \cdots, 0)$ and

$$w(x, \bar{y}) = T_x(y) - v(x).$$

as g_{i-1} is of class C^2, this is a C^1-map of D into E_n. By Lemma 15, $w(D)$ is of finite $(m + i - 1)$-extent, and hence of zero n-extent. Now let v be any vector, $\| v \| \leqq 1$, such that for some x in S, $v(x) + v$ is in $P(x)$; we shall show that v is in $w(D)$. As $P(x) = T_x(P_0)$, there is a $\bar{y}$ such that

$$v(x) + v = T_x(y), \qquad v = w(x, \bar{y});$$

as $\| v(x) + v \| \leqq K + 1$, $\| y \| \leqq K + 1$, and v is in $w(D)$. Hence there is an arbitrarily small v_0 such that no $v(x) + v_0$ is in $P(x)$; $v(x) + v_0 = \partial g_i(x, v_0)/\partial x_i$ is independent of $\partial g_{i-1}/\partial x_1, \cdots, \partial g_{i-1}/\partial x_{i-1}$ in S, and the lemma is proved.

22. **Proof of the imbedding theorem.** The last three lemmas lead at once to

LEMMA 19. *Any C^r-m-manifold M may be C^r-imbedded in E_{2m+1}.*[32]

Let $\eta(p)$ be a positive continuous function in M such that if $p_1, p_2, \cdots$ is a sequence of points of M with no limit in M, then $\lim \eta(p_i) = 0$.[33] Let f map M into the origin O in E_{2m+1}; f is of class C^r. Applying Lemma 12 with the properties of Lemma 18 gives a regular C^r-map f' approximating (f, M, r, η). We now apply Lemma 12 again, this time with the properties of Lemma 17, setting $b_0 = O$. (The new η may have to be smaller than the last.) The resulting function F is (1-1) regular and of class C^r in M, and $F(p) \neq O$ in M. By the choice of η, the limit set $LF(M)$ either is void or equals O; hence F is proper. Therefore $F(M)$ is a C^r-manifold in E_{2m+1} C^r-homeomorphic with M, and the proof is complete.

IV. THE NEIGHBORHOOD OF A MANIFOLD IN E_n

Suppose M is a C^r-m-manifold in E_n, $r \geqq 2$, $n > m$. To each point p of M there is a normal plane $P'(p)$; this family of planes fills out a neighborhood of M in E_n in a (1-1) way. If $r = 1$, this may not hold; for the $P'(p)$ depend on the first derivatives of functions defining M. Our object in this part is to find an approximating family of planes $P(p)$ of class C^r. §24 is necessary in this proof, and also directly in the next part.

23. **Projective spaces in Euclidean spaces.** The following lemma will be needed in the next §.

LEMMA 20. *Projective n-space E_n^* may be imbedded analytically in Euclidean space E_{2n+1}.*

The points of E_n^* are the sets of numbers $(x_1, \cdots, x_{n+1}) \neq (0, \cdots, 0)$, proportional sets being the same point. Let S_n be the unit n-sphere $\Sigma x_i^2 = 1$ in E_{n+1}. To each pair of "opposite" points p, $-p$ of S_n corresponds a point $\gamma(p) = \gamma(-p)$ of E_n^*. S_n is an analytic manifold with obvious neighborhoods; mapping these neighborhoods into E_n^* under γ defines E_n^* as an analytic manifold. By Lemma 19, E_n^* is C^1-homeomorphic with a manifold M' in E_{2n+1}. Let ϕ denote this homeomorphism. Then $\psi(p) = \phi(\gamma(p))$ is a (2-1) regular map of S_n into E_{2n+1}.

For any map g of S_n into E_{2n+1}, set[34]

$$Ag(p) = [g(p) + g(-p)]/2; \text{ then } A\psi(p) = \psi(p).$$

It is easily seen that if g approximates $(\psi, S_n, 1, \delta)$, then the same is true of Ag. As $Ag(p) = Ag(-p)$, we may let correspond to g a map f of E_n^* into E_{2n+1}; this

[32] Note that we may let $F(M)$ have no (finite) limit set in E_{2m+1}, by applying a transformation with reciprocal radii at the end of the proof.

[33] Let θ_i be a set of admissible maps in M. Let $\eta'(x)$ be continuous in E_m, > 0 in Q_m, and $= 0$ in $E_m - Q_m$. Set $\eta_i(p) = \eta'(\theta_i^{-1}(p))$ in U_i and $= 0$ in $M - U_i$. If $\alpha_1, \alpha_2, \cdots$ are small enough, we may set $\eta(p) = \Sigma\alpha_i\eta_i(p)$.

[34] Compare footnote[17].

might be written $f(q) = Ag(\gamma^{-1}(q))$. Take $\epsilon > 0$ such that any map approximating $(\phi, E_n^*, 1, \epsilon)$ is (1-1) regular. Choose $\delta > 0$ so that if g approximates $(\psi, S_n, 1, \delta)$, then the corresponding f approximates $(\phi, E_n^*, 1, \epsilon)$. Extend ψ through a neighborhood R of S_n in E_{n+1}, say by letting it be constant on any half ray from the origin; it is then of class C^1 in R. By Lemma 9 (or the Weierstrass approximation theorem) there is an analytic map g approximating $(\psi, R, 1, \delta)$; considering g on S_n alone, the corresponding function f has then the required properties.

24. k-**planes in** n-**space.** Let $\mathfrak{S}$ be the space whose points are the k-planes in n-space through the origin. We shall express this space as an analytic manifold $M(n, k)$ in a Euclidean space $E(n, k)$. Given the plane P, let $v_1, \cdots, v_k$ be a set of independent vectors in P; their coördinates form a matrix, with k-rowed determinants $D_{i_1 \cdots i_k}(P)$. These determinants, arranged in a sequence $D_1^*(P), \cdots, D_{\gamma_{nk}}^*(P)$, $\gamma_{nk} = \binom{n}{k}$, form the homogeneous coördinates of a point $D^*(P)$ in projective space $E_{\gamma_{nk}-1}^*$. $D^*(P)$ is independent of the vectors $v_1, \cdots, v_k$, and $D^*(P) \neq D^*(P')$ if $P \neq P'$; thus we have a (1-1) map of $\mathfrak{S}$ into a subset $\mathfrak{S}'$ of $E_{\gamma_{nk}-1}^*$. By Lemma 20, we may imbed $E_{\gamma_{nk}-1}^*$ analytically in Euclidean space $E(n, k)$; this carries $\mathfrak{S}'$ into a subset $M(n, k)$ of $E(n, k)$.

We may show that $\mathfrak{S}'$ and hence $M(n, k)$ is an analytic manifold by taking any P_0, choosing a determinant, say $D_{1 \cdots k}(P_0)$, which is $\neq 0$, and expressing each $D_{i_1 \cdots i_k}$ in terms of the determinants $D_{1, \cdots, s-1, s+1, \cdots, k, t}$ by Vahlen's relations, which are analytic; this determines maps of the required nature in $\mathfrak{S}'$. An analytically equivalent set of maps may be given as follows: Given P_0, let $p_1^0, \cdots, p_k^0$ be points of P_0 which form linearly independent vectors from the origin, and let $L_1, \cdots, L_k$ be $(n - k)$-planes (or analytic $(n - k)$-manifolds) through $p_1^0, \cdots, p_k^0$ orthogonal to P_0. If P (through the origin) is near P_0, it intersects each L_i in a point p_i; the positions of the p_i determine a map of part of $E_{k(n-k)} = E_{n-k} \times \cdots \times E_{n-k}$ (k factors) into part of $M(n, k)$.

Another important space is the space $\mathfrak{S}^*$ of all k-planes in E_n; this also forms an analytic manifold. $M(n, k)$ is closed; the present manifold is open. We may map $\mathfrak{S}^*$ into $M(n, k)$ by letting any plane correspond to the parallel plane through the origin. We shall use the symbol P for points of either space; it will always be clear which space is meant.

By an *analytic function of k-planes* we shall mean a function which, when considered in $M(n, k)$, is analytic.

We shall say two planes of any dimensions (in either space) are *orthogonal* if any vector in (or perhaps better, parallel to) one is orthogonal to any vector in the other; *independent* if they have no common vector $\neq 0$; *perpendicular* if some vector $\neq 0$ in one is orthogonal to each vector of the other. If P, P' are points of $\mathfrak{S}$, we may let $\| P' - P \|$ be the distance between the corresponding points of $M(n, k)$.

25. **The neighborhood of a manifold in space.** Two more lemmas lead up to the main result of this part, Lemma 23.

LEMMA 21. *Let M be a C^r-m-manifold in E_n ($r \geqq 1$ finite or infinite), and let $P(p)$ be a function of class C^r in M satisfying* (a) *of Lemma* 23. *Then there is a function $\xi(p)$ in M satisfying the remaining conditions.*

Take any p_0 in M. A neighborhood U of p_0 may be determined by functions (3.1) of class C^r. Define the transformation $T_{p,P}$ in terms of p_0 and $P(p_0)$ as in §19. Set $w_i(p) = T_{p,P(p)}(v_i)$; then the points of $P(p)$ for p in U are given by

$$q = p + \sum_{i=1}^{n-m} \alpha_i w_i(p). \tag{25.1}$$

Using (3.1), we may express p in terms of $y_1, \cdots, y_m$: $p = \psi(y_1, \cdots, y_m)$. Putting in (25.1) gives q as a function of $y_1, \cdots, y_m, \alpha_1, \cdots, \alpha_{n-m}$:

$$\begin{aligned} q &= \psi(y_1, \cdots, y_m) + \Sigma\alpha_i w_i(\psi(y_1, \cdots, y_m)) \\ &= g(y_1, \cdots, y_m, \alpha_1, \cdots, \alpha_{n-m}). \end{aligned} \tag{25.2}$$

g is of class C^r. Consider the vectors $\partial g/\partial y_i$, $\partial g/\partial \alpha_j$ at $q = p_0$ in U. The $\partial g/\partial y_i$ are independent vectors in the tangent plane T to M at p_0, as the α_i vanish there, and the $\partial g/\partial \alpha_j = w_j(p_0)$ are independent vectors in $P(p_0)$; as $P(p_0)$ is independent of T, the whole set of vectors is independent. In other words, the Jacobian of (25.2) is $\neq 0$ at p_0, and hence in a neighborhood R' of p_0. Solving for $y_1, \cdots, y_m$ gives p in terms of q in R': $p = H(q)$. H is of class C^r.

We may cover M by such neighborhoods R' so that any bounded closed subset of M has points in but a finite number of the R'. It is easy to construct a positive function $\xi(p)$ in M such that if $R(p)$ is that part of $P(p)$ within $\xi(p)$ of p, then $R(p)$ lies in some R'. (c) and hence (b) of Lemma 23 now hold.

LEMMA 22. *Lemma 9 holds with R and E_n replaced by analytic manifolds M and N in E_μ and E_ν respectively.*[35]

Let $P(p)$ be the normal plane to p in M. Then $P(p)$ is analytic, and hence we may define $H(p)$ by the last lemma. H is analytic. Similarly we define P' and H' for N. Set $f(q) = f(H(q))$ in $R(M)$; this is a C^r-map of $R(M)$ into the subset N of E_ν (see Lemma 4). If the analytic function F' approximates to f closely enough and $R'(M)$ is a small enough neighborhood of M in $R(M)$, then F' maps $R'(M)$ into $R(N)$, and $F = HF'$ is an analytic map of $R'(M)$ into N. F, considered on M alone, is analytic (see Lemma 3).

LEMMA 23. *Let M be a C^r-m-manifold in E_n($r \geqq 1$ finite or infinite). Then there is a positive continuous function $\xi(p)$ and a function $P(p)$ of class C^r in M, such that:* (a) *$P(p)$ is an $(n - m)$-plane through p independent of the tangent plane to M at p.* (b) *If $R(p)$ is that part of $P(p)$ within $\xi(p)$ of p, then the $R(p)$ fill out a neighborhood $R(M)$ of M in a* (1-1) *way.* (c) *If $H(q) = p$ for q in $R(p)$, then H is of class C^r in $R(M)$. Moreover, if M is analytic, so are $P(p)$ and $H(p)$.*

[35] See also Lemma 27.

We have just considered the analytic case. If $n = m$, the lemma is trivial (then H is the identity); suppose $n > m$. Let $P'(p)$ be the normal plane to M at p; there is a corresponding point $D'(p)$ in $M(n, n - m)$. D' is of class C^{r-1} and is thus continuous in M. Extended D' so as to be continuous throughout a neighborhood R of M in E_n. (Almost any method in use will do this; or we may use Lemma 11.) If R is small enough, $D'(R)$ is in $R(M(n, n - m))$; then $D'' = H'D'$ is a continuous map of R into $M(n, n - m)$, and $D'' = D'$ in M. (H' is defined in $R(M(n, n - m))$ as in the last lemma.) By Lemma 22 we may approximate D'' in R by an analytic function D so closely that if $P(p)$ is the plane through p in M parallel to the plane defined by $D(p)$, then $P(p)$ is independent of the tangent plane to M at p. $P(p)$, considered in M alone, is of class C^r, by Lemma 3. The lemma now follows from Lemma 21.

V. Analytic Manifolds

26. The lemma and method of proof. Our object in this part is to prove

Lemma 24. *Let M be a C^r-m-manifold in E_n ($r \geqq 1$ finite or infinite). Then there is a C^r-homeomorphic analytic manifold M^* in E_n.*

This, together with Lemma 19, completes the proof of Theorem I. Actually, M^*, as constructed, will approximate to M to any desired degree, but it is easier to find an approximating analytic manifold after a homeomorphic analytic one is found. (See Lemma 22.) We may suppose that $n > m$; if $n = m$, then M is analytic.

To prove the lemma, we first construct an analytic $(n - 1)$-manifold S "surrounding" M, and then find in an analytic fashion a "center" M^* of S. The proof is most easily visualized for $n = 3$, $m = 1$. The construction of S is straightforward. We determine a function positive and analytic near M and vanishing in M, subtract a very small positive analytic function, and let S be the set of points where the resulting function vanishes. The inside of S is filled up by $(n - m)$-planes $P(p)$ approximately normal to M (see Lemma 23). The resulting function $D(P)$ with values in $M(n, n - m)$ (see §24) is of class C^r inside of S. We approximate this function by an analytic function, and thus determine an analytic family of planes $P^*(p)$. (These planes, unlike the $P(p)$, intersect each other inside S.) A point p inside S is in M^* if and only if p is at the center of mass of that connected part of $P^*(p)$ inside S which contains p.

The following lemma is necessary.[36]

Lemma 25. *Given an open set R in E_n, a positive continuous function $\eta(p)$ in R, and $r \geqq 0$, there is an analytic function $\omega(p)$ in R such that*

$$(26.1) \qquad \omega(p) > 0, \qquad |D_k\omega(p)| < \eta(p) \text{ in } R \qquad (\sigma_k \leqq r).$$

Let $C_1, C_2, \cdots$ be a denumerable set of overlapping cubes covering R, and let

[36] This lemma, except for analyticity, is practically equivalent to a theorem of Ostrowski, Bull. des Sciences Math., 1934, pp. 64–72. See also our lemma 10. The theorem was known to the author in 1933. Note that we may make $r = \infty$ in a manner similar to that in AE Lemma 6.

$\phi_i(x)$ be a function of class C^∞ in R which is > 0 within C_i and $= 0$ in $R - C_i$ (see for instance AE, §9). If the a_i are small enough and positive, then $\phi(x) = \Sigma a_i \phi_i(x)$ is a positive function of class C^∞ in R satisfying the inequality (26.1). If $\omega(p)$ is an analytic function approximating (ϕ, R, r, ζ) for small enough $\zeta(p) > 0$ (see Lemma 9), then (26.1) holds.

27. **The manifold S and spheres $S^*(p, P)$.** In this section we shall find S, and shall show that certain $(n - m)$-planes P through points p near M intersect S in $(n - m - 1)$-spheres $S^*(p, P)$, which are analytic and vary analytically with p and P. In the next section we shall find the analytic manifold M^*.

Define the planes $P(p)$ and the projection $H(p)$ in the neighborhood $R(m)$ as in Lemma 23. We extend the definitions of $P(p)$ and $\xi(p)$ through $R(M)$ by setting

$$P(p) = P(H(p)), \qquad \xi(p) = \xi(H(p)).$$

Define the function $\Phi(p)$ in $R(M)$ by

$$\Phi(p) = \| p - H(p) \|. \tag{27.1}$$

As $H(p)$ is of class C^r in $R(M)$, $\Phi(p)$ is of class C^r in $R(M) - M$. By Lemmas 9 and 10 there is a function $\Phi'(p)$ continuous in $R(M)$ and analytic in $R(M) - M$ such that $\Phi' = 0$ in M, and it and its gradient satisfy

$$| \Phi'(p) - \Phi(p) | < \tfrac{1}{3}\, \xi(p), \qquad \| \nabla\Phi'(p) - \nabla\Phi(p) \| < \tfrac{1}{3} \tag{27.2}$$

in $R(M) - M$. By Lemma 25, there is a positive analytic function $\omega(p)$ in $R(M)$ such that

$$| \omega(p) | < \tfrac{1}{3}\, \xi(p), \qquad \| \nabla\omega(p) \| < \tfrac{1}{3}. \tag{27.3}$$

Set

$$\Phi^*(p) = \Phi'(p) - \omega(p); \tag{27.4}$$

then $\Phi^*(p)$ is continuous in $R(M)$ and is analytic in $R(M) - M$, and $\Phi^* < 0$ in M. S is determined by the vanishing of Φ^*.

To prove the existence of and properties of S, we shall introduce some auxiliary functions. Let p_0 be any point of M. Some neighborhood U of p_0 in M is defined by equations (3.1). Given any subset K of M, let $R(K)$ be the set of all points p of $R(M)$ such that $H(p)$ is in K. $P(p)$ is independent of T (see Lemma 23); hence there is a neighborhood U' of p_0 in M, U' in U, and a $\delta > 0$, such that if P is an $(n - m)$-plane through a point p of $R(U')$ and

$$\| P - P(p) \| < \delta, \tag{27.5}$$

then P is independent of T and hence intersects T in a unique point $H^*(P)$. (T is the tangent plane to M at p_0.) H^* is analytic. Set

$$H'(p) = H^*(P(p)), \qquad u(p) = \frac{p - H(p)}{\| p - H(p) \|}, \qquad u'(p) = \frac{p - H'(p)}{\| p - H'(p) \|}. \tag{27.6}$$

We may choose U' and δ so small that for any p in $R(U')$ and any P through p satisfying (27.5),

$$\Phi^*(H^*(P)) < 0, \qquad \| H'(p) - H(p) \| < \xi(p)/6, \tag{27.7}$$

and

$$\text{if } \Phi^*(p) \geqq 0, \quad \text{then} \quad \| u'(p) - u(p) \| < \tfrac{1}{3}. \tag{27.8}$$

For any P satisfying (27.5), let T_P be the transformation $T_{H^*(P),P}$ of §19, using the fixed point p_0 and plane $P(p_0)$, and let $S(P)$ be the unit $(n-m-1)$-sphere in P about $H^*(P)$. Given any point $\bar{q}$ of $S_0 = S(P(p_0))$, there is a corresponding point

$$q' = T_P(\bar{q}) = \mu(P, \bar{q}) \text{ in } S(P); \tag{27.9}$$

μ is analytic. To each P satisfying (27.5), each $\bar{q}$ in S_0, and any $\alpha > 0$, let correspond points p', q', q by

$$q = p' + \alpha(q' - p') = H^*(P) + \alpha[\mu(P, \bar{q}) - H^*(P)]. \tag{27.10}$$

For such values of $\alpha > 0$ which make q lie in $R(U) - U$ we define the analytic function

$$\sigma(P, \bar{q}, \alpha) = \Phi^*(q). \tag{27.11}$$

We shall show next that for some γ, $0 < \gamma < \delta$, if P is a plane through a point p of $R(U')$, $\| P - P(p) \| < \gamma$, and $\bar{q}$ is in S_0, then there is unique number

$$\alpha = \rho(P, \bar{q}) > 0 \tag{27.12}$$

which, put in (27.10) and (27.11), makes σ vanish (with q in $R(U)$); moreover, ρ is analytic. Set

$$\sigma'(p, \bar{q}, \alpha) = \sigma(P(p), \bar{q}, \alpha); \tag{27.13}$$

it is sufficient to show that, using $P(p)$, there is a unique point q of the line segment $\overline{p'q'}$ in $R(U')$ such that $\Phi^*(q) = 0$, and $\partial\sigma'/\partial\alpha > 0$ at this point.

By definition of $R(M)$, $R(U')$ contains all points of $P(p)$ within $\xi(p)$ of $H(p)$. As $p' = H'(p)$ for $P = P(p)$, (27.7) gives

$$\| p' - H(p) \| < \xi(p)/6. \tag{27.14}$$

Hence, if q'' is the point q for which $\alpha = 5\xi(p)/6$ (keeping $\bar{q}$ fixed), all of $\overline{p'q''}$ lies in $R(U')$. Moreover, as $H(q'') = H(p)$, (27.1) through (27.4) with (27.14) give

$$\Phi^*(q'') > \Phi(q'') - \tfrac{2}{3}\,\xi(p) > 0. \tag{27.15}$$

By (27.7), $\Phi^*(p') < 0$; hence there is a point of $\overline{p'q''}$ for which $\Phi^* = 0$.

Now take any q on $\overline{p'q''}$ such that $\Phi^*(q) \geqq 0$, keeping $P = P(p)$. As

$$\| q' - p' \| = 1, \qquad p' = H'(q'),$$

and $u'(q') = u'(q)$, differentiating (27.13) and using (27.10) gives

$$\frac{\partial\sigma'}{\partial\alpha} = |\nabla\Phi^*(q)\cdot(q' - p')|_{P=P(p)} = \nabla\Phi^*(q)\cdot u'(q). \tag{27.16}$$

The projection of $\nabla\Phi$ into any plane P equals the gradient of Φ as a function defined in P; hence, by (27.1) and (27.6),

$$\operatorname{Proj}_{P(p)} \nabla\Phi(q) = u(q),$$

and if u' is any vector parallel to $P(p)$, then

$$\nabla\Phi(q)\cdot u' = u(q)\cdot u'.$$

Hence, by (27.2) through (27.4) and (27.8),

$$\left|\frac{\partial\sigma'}{\partial\alpha}\right| = |\,[\nabla\Phi^*(q) - \nabla\Phi(q)]\cdot u'(q) + u(q)\cdot[u'(q) - u(q)] + 1\,| \tag{27.17}$$
$$> -\tfrac{2}{3} - \tfrac{1}{3} + 1 = 0.$$

This shows that Φ^* vanishes at a unique point q of $\overline{p'q''}$, and the existence and analyticity of ρ is proved.

Take any P satisfying (27.5) with δ replaced by γ; putting (27.12) in (27.10) gives q as a function of $\bar{q}$. As $\bar{q}$ ranges over S_0, q ranges over an analytic $(n - m - 1)$-sphere $S^*(P)$; this sphere varies analytically with P. It is the intersection of P and S. A finite or denumerable number of neighborhoods U' cover M; for each there is a corresponding $\gamma > 0$. Let $\gamma(p)$ be a positive continuous function in $R(M)$ such that $\gamma(p) = \gamma(H(p))$, and if p is in any U', then $\gamma(p)$ is less than the corresponding γ. Now if p is any point of $R(M)$, P contains p, and

$$\|\,P - P(p)\,\| < \gamma(p), \tag{27.18}$$

then $R(p)$ intersects S in an analytic sphere $S^*(p, P)$ which varies analytically with p and P.

28. **The analytic manifold** M^*. For any p in $R(M)$ and any plane P through p satisfying (27.18), let $Q^*(p, P)$ be that part of P inside $S^*(p, P)$. Let $g(p, P)$ be the center of mass of $Q^*(p, P)$. We shall show that if $P^*(p)$ is any analytic function in $R(M)$ approximating to $P(p)$ closely enough in $R(M)$, then the set M^* of points in $R(M)$ satisfying

$$g(p, P^*(p)) = p \tag{28.1}$$

is an analytic manifold in $R(M)$, C^r-homeomorphic with M.

We shall first show that $g(p, P)$ is analytic. Consider a point p_0 of M and a neighborhood U of p_0 in M etc. as before. If $V(p, P)$ is the $(n - m)$-volume of $Q^*(p, P)$ and $|\,dp\,|$ denotes the volume element, then for p in $R(U')$ and any P

through p within $\gamma(p)$ of $P(p)$,[37]

$$(28.2) \qquad g(p, P) = \frac{1}{V(p, P)} \int_{Q^*(p,P)} q \mid dq \mid.$$

We shall express this integral in a different form. The points of $Q^*(p, P)$ are given by the pairs

$$(\bar{q}, \alpha); \qquad \bar{q} \text{ in } S_0, \qquad 0 \leqq \alpha \leqq \rho(P, \bar{q}).$$

Letting W_0 be the $(n - m - 1)$-volume of S_0 and noting that T_P preserves volume, (28.2) may be written

$$(28.3) \quad g(p, P) = \frac{1}{W_0} \int_{S_0} \left[\frac{n-m}{\rho^{n-m}} \int_0^\rho \alpha^{n-m-1} \{p' + \alpha(q' - p')\} \, d\alpha \right] \mid d\bar{q} \mid,$$

where $\rho = \rho(P, \bar{q})$. This expression is easily seen to be analytic.

Let M' be the set of points in $R(M)$ satisfying $g(p, P(p)) = p$. Each $Q^*(p, P(p))$ has exactly one point in M', namely, its center of mass. Taking p_0 etc. again, set

$$(28.4) \qquad \tau(p, P) = T_P^{-1}(p) - T_P^{-1}(g(p, P)), \qquad \tau(p) = \tau(p, P(p)).$$

$\tau(p, P)$ is a point (or vector) of $E_{n-m} = P(p_0)$. $\tau(p) = 0$ if and only if p is in M'. We shall show that if $P^*(p)$ is an analytic function approximating $P(p)$ closely enough in $R(U')$ through the first order, and

$$(28.5) \qquad \tau^*(p) = \tau(p, P^*(p)),$$

then the vanishing of $\tau^*(p)$ determines an analytic manifold through p_0. To this end, let $\tau_1(p), \cdots, \tau_{n-m}(p)$ and $\tau_1^*(p), \cdots, \tau_{n-m}^*(p)$ be the components of $\tau(p)$ and $\tau^*(p)$ in the directions of fixed mutually orthogonal vectors in $P(p_0)$; then $\tau(p) = 0$ if and only if the $\tau_i(p) = 0$, and similarly for $\tau^*(p)$. The $\tau_i(p)$ vanish at a unique point of each $Q^*(p_1, P(p_1))$, and the $\nabla\tau_i(p)$ are independent as functions in $P(p_1)$; hence the same is true of the $\tau_i^*(p)$ and the $\nabla\tau_i^*(p)$, if the approximation of $P^*(p)$ is close enough. Therefore $\tau^*(p) = 0$ defines an analytic manifold M_1^* in $R(U')$, which cuts each $Q^*(p, P(p))$ in a unique point p_1', and such that the tangent plane to M_1^* at p_1' is independent of $P(p_1)$.

If $P^*(p)$ approximates to $P(p)$ closely enough in $R(M)$, then the above will hold near each point of M, and the vanishing of $\tau^*(p)$ will determine an analytic manifold M^* cutting each $Q^*(p, P(p))$ as noted. As is seen from (28.4) and (28.5), the points of M^* satisfy (28.1). The map $p' = H(p)$ of M^* into M is (1-1) and of class C^r. As the tangent plane to M^* at p is independent of $P(p)$, the inverse is also of class C^r (see Lemma 21); hence the map is a C^r-homeomorphism and the proof is complete.

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[37] Compare footnote[17].

VI. Proof of Theorem 2

We shall first prove the existence of analytic linear functionals as defined in §7; we will then be able to prove Theorem 2 and the properties in §9. Finally, we shall prove the unproved statement in Theorem 6.

29. **Real-valued analytic linear functionals.** We shall generalize AE Lemma 7 as follows:

LEMMA 26. *Let R be an open set in E, let $\Delta(p, q)$ and $\zeta(p)$ be as in §7 in R, and let r and s be finite, $s \leqq r$. Then there is an analytic linear $(R, E_1, r, \Delta, \zeta)$-functional $\mathfrak{L}$; moreover, $\mathfrak{L}$ is defined for any polynomial P of degree $\leqq s$, and $\mathfrak{L}P = P$.*

Let $R_1, R_2, \cdots$ be bounded open subsets of R such that $\bar{R}_i$ is in R_{i+1} and $R_1 + R_2 + \cdots = R$, and let ϵ_i be the minimum of $\zeta(x)$ for x in $\bar{R}_{i+1}$. Let $\Delta_i'(\rho)$ be the maximum of $\Delta(x, x')$ for points x and x' of $\bar{R}_i$ whose distance apart is ρ; these functions are easily seen to satisfy the requirements in AE Lemma 7. Let a be a fixed point of R. Given any function f of class C^r in R, set

$$\mathfrak{L}^* f(x) = \sum_{\sigma_k \leqq s} \frac{D_k f(a)}{k!} (x - a)^k; \tag{29.1}$$

This is the polynomial of degree $\leqq s$ approximating to f most closely at a. (See AE for the notation.) $\mathfrak{L}^*$ is a linear functional, and for any polynomial P of degree $\leqq s$, $\mathfrak{L}^*P = P$. As seen in AE, footnote on p. 78, for each i there is a number K_i such that if f satisfies (7.1) and hence

$$| D_k f(x') - D_k f(x) | \leqq \Delta_i'(\| x' - x \|) \text{ in } \bar{R}_i \qquad (\sigma_k \leqq s), \tag{29.2}$$

then

$$| D_k f(a) | < K_i \qquad (0 < \sigma_k \leqq s). \tag{29.3}$$

Let $\Delta_i''(\rho)$ be the maximum in $\bar{R}_i$ of

$$| D_k P(x') - D_k P(x) | \quad \text{for} \quad \| x' - x \| = \rho, \qquad \sigma_k \leqq r,$$

for polynomials $P(x)$ of degree $\leqq s$ whose derivatives at a are $\leqq K_i$, and set $\Delta_i(\rho) = \Delta_i'(\rho) + \Delta_i''(\rho)$. Now if f is any function of class C^r in R satisfying (29.2), then $f - \mathfrak{L}^*f$ satisfies the same equation with Δ_i' replaced by Δ_i.

Let $\mathfrak{L}'$ be the linear functional given by AE Lemma 7 with $M = 0$, and set

$$\mathfrak{L}f = \mathfrak{L}'(f - \mathfrak{L}^*f) + \mathfrak{L}^*f. \tag{29.4}$$

$\mathfrak{L}$ is defined for all $f = f' + P$, where f' satisfies (7.1) and hence (29.2), and P is a polynomial of degree $\leqq s$; for

$$f - \mathfrak{L}^*f = f' - \mathfrak{L}^*f',$$

and this function satisfies (29.2) with Δ_i, and is 0 at a. As both $\mathfrak{L}^*$ and $\mathfrak{L}'$ are linear, $\mathfrak{L}$ is linear. As $\mathfrak{L}'$ is analytic and $\mathfrak{L}^*f$ is a polynomial, $\mathfrak{L}$ is analytic.

Obviously $\mathfrak{L}P = P$ for polynomials P of degree $\leqq s$. Finally, $\mathfrak{L}f$ approximates (f, R, r, ζ), and the proof is complete.

30. **Analytic linear functionals.** We replace R and E_1 in the last lemma by M and E_ν, as follows.

LEMMA 27. *Let M be an analytic m-manifold in E_μ, let $\Delta(p, q)$ and $\zeta(p)$ be as in §7, and let r be finite. Then there is an analytic linear $(M, E_\nu, r, \Delta, \zeta)$-functional.*[38]

It is sufficient to prove this for real-valued functions; the general case then follows on applying it to each coördinate separately. Given any f in M, define f' in $R(M)$ by $f'(p) = f(H(p))$. We shall show that the functional of the last lemma, which we now call $\mathfrak{L}'$, may be applied to f'; then $\mathfrak{L}f$ is $\mathfrak{L}'f'$ considered in M alone.

Suppose f is of class C^r in M; then if $f'(p') = f(H(p'))$, f' is of class C^r in $R(M)$ (see the proof of Lemma 4); we let $P(p)$ be the normal plane to M at p. Differentiating $f'(p') = f'(H(p'))$ shows that $D_kf'(p')$ is a polynomial of degree $\leqq \sigma_k$ in the derivatives of order $\leqq \sigma_k$ of f' at $p = H(p')$ and of $H(p')$ at p'. Say $p = \theta_i(x)$. Then $D_kf'(p)$ is determined by the $D_sf_i(x) = D_sf(\theta_i(x))$ and the $D_t\theta_i(x)$. (The latter determine $P(p)$.) Hence

$$D_kf'(p') = \Phi[D_sf_i(x), D_t\theta_i(x), D_uH(p')] \qquad (\sigma_s, \sigma_t, \sigma_u \leqq \sigma_k), \tag{30.1}$$

for $\sigma_k \leqq r$. (k, s, t and u have respectively ν, m, m and ν components.) As the θ_i are admissible, there are but a finite number of such expressions for $D_kf'(p')$. Let a be a fixed point of M. Given any f in M or $R(M)$, set $\bar{f}(p) = f(p) - f(a)$. For any compact subset A of M there is a number K such that if f satisfies (7.1) in M, then

$$| D_k\bar{f}(p) | < K \text{ in } A \qquad (\sigma_k \leqq r), \tag{30.2}$$

(see AE, footnote on p. 78). Hence, by (30.1), for any two points p' and q' of $R(M)$ there is a number Δ such that for any such f, $(\bar{f})' = \bar{f}'$ and hence f' satisfies

$$| D_kf'(q') - D_kf'(p') | \leqq \Delta \qquad (\sigma_k \leqq r). \tag{30.3}$$

Let $\Delta^*(p', q')$ be the minimum of such numbers Δ. There are several (but a finite number of) choices for $D_kf(p)$ in (7.1); we take $\Delta^*(p', q')$ large enough for all these.

We show now that if $p'_h \to p'_0$ and $q'_h \to p'_0$, then $\Delta^*(p'_h, q'_h) \to 0$. Suppose not. Then there are a k $(\sigma_k \leqq r)$, sequences $\{p'_h\}$ and $\{q'_h\}$ approaching p'_0, and functions f_h in M satisfying (7.1), such that

$$| D_kf'_h(q'_h) - D_kf'_h(p'_h) | > \alpha > 0. \tag{30.4}$$

We may suppose that $p_0 = H(p'_0)$, $p_h = H(p'_h)$, and $q_h = H(q'_h)$ are in some U_i; (30.1) then applies. Replace f_h, f'_h by $\bar{f}_h$, $\bar{f}'_h$ as before. Then the $D_s\bar{f}_{hi}(x_h)$ are

[38] If $M = R$, we may have $\mathfrak{L}P = P$ as in the last lemma; P is a polynomial with values in E_ν if each of its coördinates is.

bounded ($x_h = \theta_i^{-1}(p_h)$), and we may suppose $D_s\bar{f}_{hi}(x_h) \to D_{is}$. As $y_h = \theta_i^{-1}(q_h) \to \lim x_h$, (7.1) shows that $D_s\bar{f}_{hi}(y_h) \to D_{is}$ also. Therefore all the variables in (30.1) approach the same limit when p is replaced by p_h as when it is replaced by q_h. As Φ is continuous, $D_k\bar{f}_h'(p_h)$ and $D_k\bar{f}_h'(q_h)$ approach the same limit; but this contradicts (30.4). It is now easy to construct a continuous function $\Delta'(p', q')$ for p', q' in $R(M)$ of the required nature, such that $\Delta'(p', q') \geqq \Delta^*(p', q')$.[39] Now if f is any function of class C^r in M satisfying (7.1), and $f'(p) = f'(H(p))$ in $R(M)$, then f' satisfies

$$(30.5) \qquad | D_k f'(q) - D_k f'(p) | \leqq \Delta'(p, q) \qquad (\sigma_k \leqq r).$$

Applying Lemma 26 to f' with $s = 0$ gives an analytic function $\mathfrak{L}'f'$ approximating $(f', R(M), r, \zeta')$; then $\mathfrak{L}f = \mathfrak{L}'f'$ in M approximates (f, M, r, ζ), if ζ' is sufficiently small.

31. **Proof of Theorem 2 with** (b) **and** (c) **omitted.** The proof of Theorem 2 with just (a) and (e′) is given by Lemma 12. We shall prove it with (a), (d), (e) and (f); the proof will be complete when we have proved (A) and (B) of §9.

We first apply Lemma 12 to find a function F' of class C^r which approximates (f, M, r, η) and has the (f, r, η)-properties $\Omega_1, \Omega_2, \cdots$. If $n \geqq 2m$, we include in these properties those of §21, to make F' regular. This is permissible, as the finiteness condition of (e) of the theorem is satisfied for these properties. Only a slight change in the proof of Lemma 18 is necessary because of E_n being replaced by N. Let W_i^* and η_i^* be the neighborhoods and functions of §7(b) and (c) corresponding to Ω_i and F'. Because of the finiteness condition, there is a positive continuous function ζ in M such that if F approximates (F', M, r, ζ), then it approximates (f, M, r, η), and for each i, it approximates $(F', \overline{W}_i^*, r, \eta_i^*)$; F then has the properties $\Omega_1, \Omega_2, \cdots$. It remains to show that the analytic function F may be chosen so as to approximate (F', M, r, ζ) and have the properties $\Omega_1', \Omega_2', \cdots$.[40]

Replace the $\Delta(p, q)$ of the theorem if necessary by a larger Δ so that (7.1) is satisfied with f' and Δ replaced by F' and $\frac{1}{2}\Delta$. Let $\mathfrak{L}$ be the analytic linear $(M, E_\nu, r, \Delta, \frac{1}{2}\zeta)$-functional given by Lemma 27; we suppose ζ is so small that if F'' approximates (F', M, r, ζ), then $F''(M)$ is in $R(N)$. For some ζ', if F'' satisfies (7.1) and approximates (F', M, r, ζ'), then $\mathfrak{L}F''$ approximates (F', M, r, ζ). We must now choose F'' so that it approximates (F', M, r, ζ') and satisfies (7.1), and so that $F = \mathfrak{L}F''$ has the properties $\Omega_1', \Omega_2', \cdots$.

---

[39] Let $\rho(p)$ be the smaller of 1 and half the distance from p to $E_\nu - R(M)$, and let $\rho(p, q)$ be the smaller of $\rho(p)$, $\rho(q)$, $\| q - p \|$. Take p and q in $R(M)$, let $\delta_\alpha(p, q)$ be the upper bound of $\Delta^*(p', q')$ for $\| p' - p \| \leqq \alpha$, $\| q' - q \| \leqq \alpha$, and set

$$\Delta'(p, q) = \| q - p \| + \frac{1}{\rho(p, q)} \int_0^{\rho(p, q)} \delta_\alpha(p, q)\, d\alpha.$$

[40] *If* $n \geqq 2m + 1$ *and* f *is proper*, we may find an F which is analytic, regular, (1-1) and proper, and has the properties $\Omega_1, \Omega_2, \cdots$, by including in these properties those of §§20 and 21, and then applying Lemma 22 with its $\eta(p)$ sufficiently small. (See the end of §6.)

Let W_i, W_i', G_j^i be the open sets and functions corresponding to Ω_i' ($i = 1, 2, \cdots$). We shall choose sets of numbers α^i so that if

$$f_0 = F', \qquad f_i = f_{i-1} + \sum_j \alpha_j^i G_j^i, \qquad F'' = \lim f_i,$$

then F'' is the required function. As the W_i' are bounded, if f' is a function satisfying an inequality of the nature of (7.1), and α is small enough, then $f' + \sum_{i,j} \alpha_j^i G_j^i$ approximates to f' as closely as we please and satisfies (7.1) with a new Δ as near the old as we wish. Hence we may choose numbers $\bar{\alpha}^1, \bar{\alpha}^2, \cdots$ such that if $\alpha^1, \alpha^2, \cdots$ is any sequence with $|\alpha_j^i| \leqq \bar{\alpha}^i$, then F'', using these α^i, satisfies (7.1) and approximates (F', M, r, ζ'). The proof now runs exactly like that of Lemma 12, except that $H\mathfrak{L}f_j$ ($j \geqq i$) and $H\mathfrak{L}F''$ will have the property Ω_i'. The existence of α^i at each step such that $H\mathfrak{L}f_i$ has the property Ω_i is given by (d′) of §7.

32. **Certain maps of M into N like translations near a point.** Let f be a C^r-map of M into $N(r \geqq 0)$, take p_0 in M, and set $q_0 = f(p_0)$. Given any $\delta > 0$, we shall find neighborhoods W', W of p_0 in M such that $\overline{W}'$ is in W and $\overline{W}$ is within δ of p_0 (measuring in E_μ), and we shall find C^r-maps $G_1, \cdots, G_n$ of M into E_ν such that $G_i(p) = O$ for p in $M - W$, and such that

$$(32.1) \qquad f_\alpha(p) = H\mathfrak{L}[f(p) + \sum \alpha_i G_i(p)] = H[\mathfrak{L}f(p) + \sum \alpha_i \mathfrak{L}G_i(p)]$$

for $|\alpha_i| \leqq 1$ is an n-parameter family of C^r-maps in $\overline{W}'$. $\mathfrak{L}$ is any $(M, E_\nu, r, \Delta, \zeta)$-functional for Δ large enough and ζ small enough near p_0.

Let θ be a C^r-map of Q_m into a neighborhood W of p_0; we will determine the size of W later. Set $W' = \theta(Q_m')$. Let $\lambda(p)$ be of class C^r in M, $= 1$ in $\overline{W}'$, and $= 0$ in $M - W$ (see §20). Let P be the tangent plane to N at q_0, let $y_1, \cdots, y_\nu$ be rectangular axes in E_ν such that P is the $(y_1, \cdots, y_n)$-plane, and let v_i be the unit vector in the direction of y_i. Set

$$(32.2) \qquad G_i(p) = \lambda(p)v_i \text{ in } M \qquad (i = 1, \cdots, n).$$

In using Lemma 23, let $P(q)$ be the normal plane to q in N. Then obviously

$$(32.3) \qquad \frac{\partial H(q_0)}{\partial y_j} = v_j\ (j = 1, \cdots, n), \text{ and } = 0 \qquad (j = n+1, \cdots, \nu).$$

Putting (32.2) in (32.1) and differentiating gives therefore

$$(32.4) \qquad \frac{\partial f_\alpha(p_0)}{\partial \alpha_i} = \sum_{j=1}^{n} [\mathfrak{L}G_i(p_0)]_j v_j = \text{Proj } \mathfrak{L}G_i(p_0),$$

where Proj v is the projection of a vector v in E_ν into P. If we leave out $\mathfrak{L}$ in (32.4), we may choose W so that the resulting vectors $\partial f_\alpha / \partial \alpha_i$ are independent in $\overline{W}'$. We then choose Δ and ζ so that $\mathfrak{L}$ is defined for the terms in (32.1) with $|\alpha_i| \leqq 1$, and so that the vectors $\partial f_\alpha / \partial \alpha_i$ are independent in $\overline{W}'$; then $f_\alpha(p)$ *is an n-parameter family in $\overline{W}'$.*

33. **Certain maps of M into N like rotations plus translations near a point.** Let f be a C^r-map ($r \geqq 1$) of M into N. Take p_0, q_0, etc. as before; we shall choose W, W', and C^r-maps G_{ij}. Let $\bar{E} = E_{(m+1)n}$ be a Euclidean space with coördinates

$$z_{ij} \qquad (i = 1, \cdots, m+1; j = 1, \cdots, n).$$

The subscripts i, j will, in this section, always range over the values shown, unless otherwise stated. Let $\bar{v}_{ij}$ be the unit vector in the direction of z_{ij}. Let θ map Q_m into W. We may consider $(x_1, \cdots, x_m)$ (in Q_m) as coördinates in W, and write

$$x \text{ for } p = \theta(x), \qquad \frac{\partial g(p)}{\partial x_i} \text{ for } \left.\frac{\partial g(\theta(x))}{\partial x_i}\right|_{x=\theta^{-1}(p)} \text{ in } W.$$

Corresponding to any C^r-map g of W into E_ν, define the C^{r-1}-map $\bar{g}$ of W into $\bar{E}$ by

$$\bar{g}(p) = \sum_{j;\, i \leqq m} \frac{\partial g_j(p)}{\partial x_i} \bar{v}_{ij} + \sum_j g_j(p)\bar{v}_{m+1,j}; \tag{33.1}$$

g_j is the j^{th} component of g in E_ν. (Note that j runs to n only.) Define

$$G'_{ij}(p) = \lambda(p)x_i v_j \ (i \leqq m), \qquad G'_{m+1,j}(p) = \lambda(p)v_j. \tag{33.2}$$

For these G, (33.1) gives in $\overline{W}'$

$$\bar{G}'_{ij}(p) = \bar{v}_{ij} + x_i\bar{v}_{m+1,j} \ (i \leqq m), \qquad \bar{G}'_{m+1,j}(p) = \bar{v}_{m+1,j}. \tag{33.3}$$

These vectors are obviously independent for p in $\overline{W}'$.

The family of maps $f_\beta(p)$ will be defined by

$$f_\beta(p) = H\mathfrak{L}[f(p) + \sum_{i,j} \beta_{ij}G'_{ij}(p)]. \tag{33.4}$$

As before, we find

$$\frac{\partial}{\partial \beta_{ij}} f_\beta(p_0) = \text{Proj } \mathfrak{L}G'_{ij}(p_0).$$

As $\text{Proj } G'_{ij}(p) = G'_{ij}(p)$, the vectors $\overline{\text{Proj } \mathfrak{L}G'_{ij}(p_0)}$ are linearly independent for small enough ζ. It is easily seen that the operations of passing from g to $\bar{g}$ and of differentiating are permutable; hence, as before, we may take W and W', Δ, and ζ so that the vectors $\partial\bar{f}_\beta(p)/\partial\beta_{ij}$ are linearly independent in $\overline{W}'$. Hence $\bar{f}_\beta$ *is an* $(m+1)n$*-parameter family of* C^{r-1}*-maps of* $\overline{W}'$ *into* $\bar{E}$.

34. **Proof of** (B) **and** (C), §9. In the hypothesis of (B), $n \geqq 2m+1$. As regularity is taken care of by (e) of the theorem, we may first replace the given map by a regular C^r-map; let the new map be f. As f is locally (1-1), we may find a positive continuous function $\delta(p)$ in M such that $f(p)$ is (1-1) for p within $\delta(p_0)$ of p_0, for any p_0. For each p_0 in M there is a neighborhood W as in §32; moreover, these may be taken arbitrarily small. Hence we may choose such

neighborhoods $W_1, W_2, \cdots$ such that $W_1' + W_2' + \cdots$ covers M, any compact subset of M has points in common with but a finite number of the $\overline{W}_s$, and if p is in $\overline{W}_s' + \overline{W}_t'$, $\overline{W}_s \cdot \overline{W}_t \neq 0$, then $\overline{W}_s + \overline{W}_t$ lies within $\delta(p)$ of p. Next define the maps $G_1^s, \cdots, G_n^s$ $(s = 1, 2, \cdots)$ as in §32. We may suppose that the η of the theorem is so small that any f' approximating $(f, M, 1, \eta)$ is (1-1) in all $\overline{W}_s + \overline{W}_t$ with $\overline{W}_s \cdot \overline{W}_t \neq 0$.

Arrange the pairs of numbers (s, t) for which $\overline{W}_s \cdot \overline{W}_t = 0$ in a sequence. For any k, let (s, t) be the k^{th} member of the sequence, and let Ω_k be the property of maps f' which holds if $f'(\overline{W}_s')$ does not intersect $f'(\overline{W}_t')$. The property of f' of being (1-1) is the sum of these properties. We shall show that Ω_k is an $[f, r, \eta, \Delta, \zeta]$-property. (a), (b) and (c) of §7 hold; we shall prove (d′), with W, W', G_i replaced by $W_s + W_t$, $W_s' + W_t'$, G_i^s. Given f' and $\mathfrak{L}$, set

$$f_{\alpha^s}'(p) = H\mathfrak{L}\left[f'(p) + \sum_{i=1}^{n} \alpha_i^s G_i^s(p)\right]. \tag{34.1}$$

Applying §32, we now see that f_{α^s}' is an n-parameter family of C^r-maps $(r \geqq 1)$ in $\overline{W}_s'$. As $f'(\overline{W}_t')$ is of zero $(m + 1)$-extent in N, $\overline{W}_s'$ is of finite m-extent in M, $(m + 1) + m \leqq n$, and $f_{\alpha^s}' = f'$ in W_t, there is an arbitrarily small α^s such that $f_{\alpha^s}(\overline{W}_s') \cdot f_{\alpha^s}(\overline{W}_t') = 0$ (Lemma 16), and (d′) is proved.

To prove (C), §9 we proceed as above. Let $K = K_1 + K_2 + \cdots$ be the subset of N, each K_s being of zero $(n - m)$-extent. Arrange the pairs (s, t) in a sequence, and let Ω_k hold if $f'(\overline{W}_s')$ does not intersect K_t. The proof runs now exactly as above.

35. **Proof of** (A) **and** (D), §9. As before, we may suppose that the given map is regular. If $r = 0$ or 1, we may at the beginning replace the map f by a C^2-map (Lemma 22). Hence we suppose that $r \geqq 2$. Define $\delta(p)$ and the W_s, W_s' exactly as in §34, and define the G_{ij}' as in §33. Again, let k correspond to (s, t), and let Ω_k be the property of maps f' which holds if f' has at most regular singularities in $\overline{W}_s' + \overline{W}_t'$; complete regularity is the sum of these properties. We must prove (d′) for Ω_k.

Before proceeding, consider (D). Let $V_1, V_2, \cdots$ be admissible neighborhoods in N', and let Ω_k^* hold if $\overline{W}_s'$ intersects $\bar{V}_t$ only in the proscribed manner. If $n' = m$, this is the same as stating that $f'(\overline{W}_s') + \bar{V}_t$ has at most regular singularities. If we show how to transform $\overline{W}_s'$ with reference to $\bar{V}_t$, the same process transforms $\overline{W}_s'$ with reference to $\overline{W}_t'$; hence we need merely prove (d′) for (D).

Given f' and $\mathfrak{L}$, set

$$f_{\beta^s}'(p) = H\mathfrak{L}\left[f'(p) + \sum_{i,j} \beta_{ij}^s G_{ij}'^s(p)\right], \tag{35.1}$$

To each f' corresponds an $(m + 1)n$-parameter family of maps $\bar{f}_{\beta^s}'$ of $\overline{W}_s'$ into $\bar{E}$, by §33. We shall show that if $\bar{f}_{\beta^s}'(\overline{W}_s')$ avoids a certain set S, then $f_{\beta^s}'(\overline{W}_s')$ intersects $\bar{V}_t$ in the proper manner. S will be the sum of a denumerable num-

ber of sets of finite $(m + 1)(n - 1)$-extent. As the number of parameters in f'_{β^s} is $(m + 1)n$ and the dimension of W_s is m, we may apply Lemma 16 and make $\overline{W}'_s$ avoid any of these. Applying the process in the proof of Lemma 12, we may make $\overline{W}'_s$ avoid S, and thus complete the proof.

We shall leave out the indices s, t in what follows. The vector $\partial' f'_\beta(p)/\partial x_i$ in P (see §32) with components $\partial f'_{\beta j}(p)/\partial x_i$ $(j = 1, \cdots, n)$, by definition, (see §33) is the projection of the vector $\partial f'_\beta(p)/\partial x_i$ in E_ν into P. As f'_β is regular (for small β^s_{ij}), the latter vectors $(i = 1, \cdots, m)$ are independent. As the W'_s may be taken arbitrarily small, we may suppose that the $\partial' f'_\beta(p)/\partial x_i$ are independent in $\overline{W}'$. As N' may be cut into arbitrarily small pieces, we may suppose that $\bar{V}$ projects in a (1-1) regular manner into P (if $\bar{V}$ intersects $f'_\beta(\overline{W}')$ for any β). As $f'_\beta(\overline{W}')$ and $\bar{V}$ are both in N, and the projection of N into P is regular near q_0, they intersect in an allowable manner in N if and only if their projections do in P. Let $P\bar{V}$, Pq etc. denote the projections of $\bar{V}$, q, etc. into P.

Let p be a point of $\overline{W}'$, and q, a point of $\bar{V}$; we shall consider under what conditions $Pf'_\beta(p) = Pq$, the intersection being of an unallowable character. This is so if the vectors $\partial' f'_\beta(p)/\partial x_i$ determine a plane P_m in P which has a plane P_h of dimension $h > k = m + n' - n$ in common with the plane $P_{n'}$ tangent to $P\bar{V}$ at $q' = Pq$. Hence the set S in $\bar{E}$ which $\bar{f}'_\beta(\overline{W}')$ must avoid is the set of points z with the following property. For some q in $\bar{V}$, some $h > k$ and plane P_h in the tangent plane $P_{n'}$ to $P\bar{V}$ at $q' = Pq$, and some plane P_m which contains q' and has exactly P_h in common with $P_{n'}$, the last n coördinates of z determine (in P) the point q' and the first mn coördinates determine the direction of P_m. Let z_0 be that point of S we have just described; we shall consider that part of S_h near z_0, S_h being those points of S with this corresponding h.

A point z of S_h is determined by the set

$$(q, P_h, P_m, v), \quad \text{where} \quad v = (v_1, \cdots, v_m), \qquad v_i = \frac{\partial' f'_\beta(p)}{\partial x_i},$$

q, P_h, P_m, v being chosen in the order given. (The last n coördinates of z are then determined.) q runs over a set of dimension n'. Now keep q fixed, and vary P_h. P_h lies in $P_{n'}$ and contains q'; the dimension of such a set of planes is $h(n' - h)$. Next vary P_m. It is determined by naming a plane P_{m-h} through q' in the plane P_{n-h} through q' orthogonal to P_h; hence the set of planes P_m is of dimension $(m - h)(n - m)$. Finally, vary the vectors v_i. Each one may vary freely as long as it remains in P_m; hence the dimension of this set of positions of the set of vectors is m^2. Consequently, if we set $h = k + h'$ and note that $n' - h = n - m - h'$, the above set runs over a part of a Euclidean space of dimension

$$\begin{aligned} d_h &= n' + h(n' - h) + (m - h)(n - m) + m^2 \\ &= n' - hh' + (h + m - h)(n - m) + m^2 \\ &= mn + n' - hh' \leqq mn + n' - h \leqq mn + n - m - 1. \end{aligned}$$

The map of this set into the corresponding part of S_h is of class C^1, and hence the $(m + 1)(n - 1)$-extent of the latter is finite, as required.

36. **Completion of the proof of Theorem** 6. Let $\theta_1, \theta_2, \cdots$ be a completely admissible set of maps in that part M^* of M' for which $0 < t < 1$. We take them so that if $\overline{U}_i \cdot \overline{U}_j \neq 0$, then f is (1-1) in $\overline{U}_i + \overline{U}_j$. In applying Theorem 2 in the proof of Theorem 6, let us introduce (f, r, η)-properties as follows. Arrange the pairs of numbers (i, j) in a sequence. Let the k^{th} member be (i, j). Then Ω_k holds for $f(p \times t) = \phi_t(p)$ if the following is true. If $0 < t < 1$, $p \times t$ is in $\overline{U}'_i$, and $q \times t$ is in $\overline{U}'_j$, then $f(p \times t) \neq f(q \times t)$. We shall show that each property is an (f, r, η)-property (for small enough η). It will follow that *each* $\phi_t(0 < t < 1)$ *may be made* (1-1), *irrespective of* M_0 *and* M_1. The new map of M' into N is of class C^r, by Lemma 10.

We may suppose the U_l are so small that if $\overline{U}_i \cdot \overline{U}_j \neq 0$, then $\overline{U}_i + \overline{U}_j$ is in some V_h in N, and so that vectors $v_1, \cdots, v_{n-m}$ may be chosen in E_n with the following property. Set $f'_h(p \times t) = \chi_h^{-1} f(p \times t)$, and let $U'_j(t)$ be the set of points $p \times t$ of U'_j whose second coördinate is t. If P and P' are the m- and $(n - m)$-planes through the origin in E_n, the first being orthogonal to and the second parallel to $v_1, \cdots, v_{n-m}$, then any $f'_h(\overline{U}_j(t))$ projects into a subset of P so that both the projection and its inverse are of class C^r. The points

$$q = p + \sum_{s=1}^{n-m} \alpha_s v_s, \qquad p \text{ in } f'_h(\overline{U}_j(t)),$$

fill out an open set R_t in E_n, and if solving this (see §3) gives

$$p = H(q, t), \qquad \alpha_s = \Phi_s(q, t),$$

then H and the Φ_i are of class C^r. To any C^r-map g of U_i into Q_n let $\bar{g}$ be the map of U_i into E_{n-m} whose s^{th} component is

$$\bar{g}_s(p \times t) = \Phi_s(g(p \times t), t).$$

Define the family of maps

$$g_\beta(p \times t) = f'_h(p \times t) + \sum_{s=1}^{n-m} \beta_s v_s \text{ in } \overline{U}_i;$$

then the $\bar{g}_\beta$ have the property that

$$\frac{\partial}{\partial \beta_s} \bar{g}_\beta(p \times t) = v_s.$$

Hence the $\bar{g}_\beta$ form an $(n - m)$-parameter family in E_{n-m}. As $\overline{U}_i$ is of dimension $m + 1 < n - m$, there is an arbitrarily small β such that $\bar{g}_\beta(p \times t) \neq 0$ in $\overline{U}_i$ (see Lemmas 14–16). For this β, no $g_\beta(\overline{U}'_i(t))$ intersects any $f'_h(\overline{U}'_j(t))$ (same t), for no $\Phi_s(g_\beta(p \times t), t) = 0$. Consequently, using $\lambda(p)$ etc. as in §20, we prove (d) of §7. The other properties are obvious, and the statement is proved.

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MULTIPLICATIONS ON A COMPLEX

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In their communications at the First International Topological Conference (Moscow, September 1935), J. W. Alexander and A. Kolmogoroff introduced the notion of a dual cycle[1] and defined a product of a dual p-cycle and a dual q-cycle, this product being a dual $(p + q)$-cycle. A different multiplication of the same sort is considered in this paper. It may be shown that the Alexander-Kolmogoroff product, augmented by the dual boundary of a suitable $(p + q - 1)$-chain, is equal to the $\binom{p+q}{p}^{\text{th}}$ multiple of the product here introduced.[2] Moreover, I consider also a product of an ordinary n-cycle and a dual p-cycle $(n \geqq p)$, this product being an ordinary $(n - p)$-cycle. There is a simple algebraic relationship between the two kinds of multiplication, which I shall explain elsewhere. As an application of the general theory, I give a new approach to the duality and intersection theory of a combinatorial manifold, given in a simplicial subdivision. The theory works exclusively in the given subdivision.

This is a preliminary paper of a purely combinatorial nature. In a later paper, I shall apply the same methods to general topological spaces, and in particular to the very general "manifolds" defined in my recent note in the *Proceedings of the National Academy of Sciences* (U. S. A.).

Many of the results of this paper were found independently by H. Whitney, but his methods of proof seem not much related to mine.

1. Let there be given a complex K. We shall designate by σ_i^p $(p = 0, 1, 2, \cdots)$ the (oriented) p-simplices of K and by $\eta_{i\,j}^{p}$ $(= 0, 1, -1)$ the incidence coefficient of σ_i^{p+1} and σ_j^p.

The word *group* will always designate an additively written abelian group. If $\mathfrak{A}$ is a group, then a $(p, \mathfrak{A})$-chain is a symbol of the form $a_i\sigma_i^p$, $a_i \in \mathfrak{A}$, where, as always in this paper, one has to sum over every subscript appearing twice.

The *boundary* FA^p of a $(p, \mathfrak{A})$-chain $A^p = a_i\sigma_i^p$ is zero if $p = 0$, and it is the $(p - 1, \mathfrak{A})$-chain

$$FA^p = \eta_{i\,j}^{p-1} a_i \sigma_j^{p-1}$$

[1] As a matter of fact the *Topology* of S. Lefschetz (1930), contains an essentially equivalent notion (pp. 282–286).

[2] In his paper "On the Connectivity Ring of an Abstract Space" in this number of the *Annals of Mathematics*, pp. 698–708, J. W. Alexander has modified his definition, and it is now in agreement with the one here presented.

if $p > 0$. If $FA^p = 0$, we say that A^p is an *ordinary* $(p, \mathfrak{A})$-*cycle*. The $(p, \mathfrak{A})$-chain FA^{p+1} is an ordinary $(p, \mathfrak{A})$-cycle for every $(p + 1, \mathfrak{A})$-chain A^{p+1}. Two ordinary $(p, \mathfrak{A})$-cycles A_1^p and A_2^p are said to be of the same homology class, or to be homologous to each other (in symbols $A_1^p \sim A_2^p$) if there exists a $(p + 1, \mathfrak{A})$-chain A_0^{p+1} such that

$$A_1^p - A_2^p = FA_0^{p+1}.$$

The *dual boundary* F^*A^p of a $(p, \mathfrak{A})$-chain $A^p = a_i\sigma_i^p$ is the $(p + 1, \mathfrak{A})$-chain

$$F^*A^p = \eta_{ji}^p a_i \sigma_j^{p+1}.$$

If $F^*A^p = 0$, we say that A^p is a *dual* $(p, \mathfrak{A})$-*cycle*. The $(p + 1, \mathfrak{A})$-chain F^*A^p is a dual $(p + 1, \mathfrak{A})$-cycle for every $(p, \mathfrak{A})$-chain A^p. Two dual $(p, \mathfrak{A})$-cycles A_1^p and A_2^p are said to be of the same homology class, or to be homologous to each other (in symbols $A_1^p \sim A_2^p$) (1) in the case $p = 0$ only if they are identical, (2) in the case $p > 0$ if there exists a $(p - 1, \mathfrak{A})$-chain A_0^{p-1} such that

$$A_1^p - A_2^p = F^*A_0^{p-1}.$$

2. Let $\mathfrak{B}$ be a given group. Let B^q be a given dual $(q, \mathfrak{B})$-cycle. By an *auxiliary construction* we mean an operation attaching to every simplex σ_i^p $(p = 0, 1, 2, \cdots)$ a $(p + q, \mathfrak{B})$-chain $B^{p+q}(\sigma_i^p)$ such that the following three conditions are satisfied. *First*, if the coefficient of a $(p + q)$-simplex τ^{p+q} in $B^{p+q}(\sigma_i^p)$ is different from zero, then σ_i^p must be a face of τ^{p+q}. *Second*, we must have

$$B^q = \sum_i B^q(\sigma_i^0). \tag{2.1}$$

Third, we must have for every simplex σ_i^p $(p = 0, 1, 2, \cdots)$

$$F^*B^{p+q}(\sigma_i^p) = \sum_j \eta_{ji}^p B^{p+q+1}(\sigma_j^{p+1}). \tag{2.2}$$

3. We shall prove that *the auxiliary construction is always possible*. Let there be given a fixed ordering of the vertices of K. Let σ^p be a given p-simplex, written as

$$\sigma^p = (v_0, v_1, \cdots, v_p)$$

corresponding to the given ordering of vertices (i.e. v_0 precedes v_1 etc.). We shall define the $(p + q, \mathfrak{B})$-chain $B^{p+q}(\sigma^p)$ as follows. The only $(p + q)$-simplices appearing in $B^{p+q}(\sigma^p)$ will have, corresponding to the given ordering of vertices, the form

$$(v_0, v_1, \cdots, v_p, \cdots, v_{p+q}), \tag{3.1}$$

i.e. the first $p + 1$ vertices will be those of σ^p. The coefficient of any such simplex (3.1) in $B^{p+q}(\sigma^p)$ will be equal to the coefficient of the q-simplex $(v_p, \cdots, v_{p+q})$ in B^q.

The first two properties of the auxiliary construction being evident, we have only to prove (2.2) for

$$\sigma_i^p = \sigma^p = (v_0, v_1, \cdots, v_p).$$

The only $(p + q + 1)$-simplices τ^{p+q+1} appearing on either side of (2.2) must all have σ^p as their common face and, moreover, corresponding to the given ordering of vertices, the vertex v_p must be either the $(p + 1)^{\text{th}}$ or the $(p + 2)^{\text{th}}$ vertex of τ^{p+q+1}. We have to prove that any such τ^{p+q+1} has equal coefficients on both sides of (2.2). This being quite evident in the case where v_p is the $(p + 2)^{\text{th}}$ vertex of τ^{p+q+1}, we only have to examine the case when, in the given order of vertices, we have

$$\tau^{p+q+1} = (v_0, v_1, \cdots, v_p, \cdots, v_{p+q+1}).$$

Let b_{p+i} $(0 \leqq i \leqq q + 1)$ be the coefficient, in the $(q, \mathfrak{B})$-chain B^q, of the oriented q-simplex obtained from $(v_p, \cdots, v_{p+q+1})$ by omitting the vertex v_{p+i}. The coefficients of τ^{p+q+1} in both sides of (2.2) are respectively equal to

$$(-1)^{p+1}b_{p+i} \text{ and to } (-1)^{p+1}b_p.$$

But since B^q is a dual $(q + 1, \mathfrak{B})$-cycle, the coefficient of the $(q + 1)$-simplex $(v_p, \cdots, v_{p+q+1})$ in F^*B^q must vanish, i.e.

$$(-1)^i b_{p+i} = 0 \quad \text{or} \quad (-1)^{p+i}b_{p+i} = (-1)^{p+1}b_p.$$

4. Let us suppose that the dual $(q, \mathfrak{B})$-cycle B^q is identically zero. $B^{p+q}(\sigma_i^p)$ being the elements of an auxiliary construction chosen in any manner corresponding to $B^q = 0$, we shall prove that we may attach to every p-simplex $\sigma_i^p (p = 1, 2, 3, \cdots)$ a $(p + q - 1, \mathfrak{B})$-chain $C^{p+q-1}(\sigma_i^p)$ such that the following three conditions are satisfied. *First,* if the coefficient of a $(p + q - 1)$-simplex τ^{p+q-1} in $C^{p+q-1}(\sigma_i^p)$ is different from zero, then σ_i^p must be a face of τ^{p+q-1}. *Second,* we have for every 0-simplex σ_i^0

$$B^q(\sigma_i^0) = \eta_{ji}^0 C^q(\sigma_j^1). \tag{4.1}$$

Third, we have for every p-simplex σ_i^p, where $p = 1, 2, 3, \cdots$,

$$B^{p+q}(\sigma_i^p) = \eta_{ji}^p C^{p+q}(\sigma_j^{p+1}) + F^*C^{p+q-1}(\sigma_i^p). \tag{4.2}$$

We begin by the construction of $C^q(\sigma_j^1)$. Let τ^q be any q-simplex and let $b_i(\tau^q)$ be its coefficient in $B^q(\sigma_i^0)$. If σ_i^0 is not a vertex of τ^q, we have $b_i(\tau^q) = 0$. Moreover, since $B^q = 0$, it follows from (2.1) that $\sum_i b_i(\tau^q) = 0$. Therefore, $b_i(\tau^q) \cdot \sigma_i^0$ is an ordinary (0, $\mathfrak{B}$)-cycle of the q-simplex τ^q having zero as the sum of its coefficients. It is well known that such a (0, $\mathfrak{B}$)-cycle is equal to the boundary of a (1, $\mathfrak{B}$)-chain of the q-simplex τ^q. Therefore there exists, for every 1-simplex σ_j^1, an element $c_j(\tau^q)$ of the group $\mathfrak{B}$ such that (1) $c_j(\tau^q) = 0$ if σ_j^1 is not a face of τ^q, (2) $b_i(\tau^q) = \eta_{ji}^0 c_j(\tau^q)$ for every σ_i^0. Let us put

$$C^q(\sigma_j^1) = \sum c_j(\tau^q)\,\tau^q,$$

the summation running over all q-simplices τ^q. Then σ_j^1 is a face of every q-simplex appearing in $C^q(\sigma_j^1)$ and the relations (4.1) hold true.

If we put $C^{q-1}(\sigma_i^0) = 0$, the relation (4.2) corresponding to $p = 0$ reduces to (4.1). Therefore, we may suppose our construction carried through up to the relations (4.2), where p is given, and we have to construct $(p + q + 1)$-chains $C^{p+q+1}(\sigma_k^{p+2})$ satisfying the analogous relations

$$B^{p+q+1}(\sigma_j^{p+1}) = \eta_{kj}^{p+1} C^{p+q+1}(\sigma_k^{p+2}) + F^*C^{p+q}(\sigma_j^{p+1}). \tag{4.3}$$

Since $F^*C^{p+q-1}(\sigma_i^p)$ is a dual $(p + q, \mathfrak{B})$-cycle, it follows from (4.2) that

$$F^*B^{p+q}(\sigma_i^p) = \eta_{ji}^p F^*C^{p+q}(\sigma_j^{p+1}).$$

Comparing with (2.2) we get

$$\eta_{ji}^p B^{p+q+1}(\sigma_j^{p+1}) - F^*C^{p+q}(\sigma_j^{p+1}) = 0. \tag{4.4}$$

Now let τ^{p+q+1} be any $(p + q + 1)$-simplex and let $b_j(\tau^{p+q+1})$ be its coefficient in the $(p + q + 1, \mathfrak{B})$-chain

$$B^{p+q+1}(\sigma_j^{p+1}) - F^*C^{p+q}(\sigma_j^{p+1}).$$

If σ_j^{p+1} is not a face of τ^{p+q+1}, we have $b_j(\tau^{p+q+1}) = 0$. Moreover, it follows from (4.4) that $\eta_{ji}^p b_j(\tau^{p+q+1}) = 0$. Therefore, $b_j(\tau^{p+q+1})\sigma_j^{p+1}$ is an ordinary $(p + 1, \mathfrak{B})$-cycle of the $(p + q + 1)$-simplex τ^{p+q+1}. It is well known that such a $(p + 1, \mathfrak{B})$-cycle is equal to the boundary of a $(p + 2, \mathfrak{B})$-chain of the simplex τ^{p+q+1}. Therefore there exists, for every $(p + 2)$-simplex σ_k^{p+2}, an element $c_k(\tau^{p+q+1})$ of the group $\mathfrak{B}$ such that (1) $c_k(\tau^{p+q+1}) = 0$ if σ_k^{p+2} is not a face of τ^{p+q+1}, (2) $b_j(\tau^{p+q+1}) = \eta_{kj}^{p+1} c_k(\tau^{p+q+1})$ for every σ_j^{p+1}. Let us put

$$C^{p+q+1}(\sigma_k^{p+2}) = \sum c_k(\tau^{p+q+1}) \cdot \tau^{p+q+1}$$

the summation running over all $(p + q + 1)$-simplices τ^{p+q+1}. Then σ_k^{p+2} is a face of every $(p + q + 1)$-simplex appearing in $C^{p+q+1}(\sigma_k^{p+2})$ and the relations (4.3) hold true.

5. Let there be given three groups $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$. Let there be given a law attaching to every couple a, b, where $a \epsilon \mathfrak{A}$ and $b \epsilon \mathfrak{B}$, an element $c \epsilon \mathfrak{C}$, called the *product* of a and b and designated by ab or $a \cdot b$. Furthermore, let us suppose the validity of the distributive laws

$$(a_1 + a_2)b = a_1b + a_2b, \qquad a(b_1 + b_2) = ab_1 + ab_2.$$

In such circumstances, we put $\mathfrak{C} = (\mathfrak{A}, \mathfrak{B})$ and say that there is given an $(\mathfrak{A}, \mathfrak{B})$-*multiplication.*

Any $(\mathfrak{A}, \mathfrak{B})$-multiplication defines an "*inverse*" $(\mathfrak{B}, \mathfrak{A})$-multiplication (with the same group $\mathfrak{C}$), if we define the new product ba to be equal to the original product ab.

6. Let there be given an $(\mathfrak{A}, \mathfrak{B})$-multiplication. Let

$$A^p = a_i \sigma_i^p$$

be a dual $(p, \mathfrak{A})$-cycle. Let B^q be a dual $(q, \mathfrak{B})$-cycle. We shall define their product $A^p B^q$ as a dual $[p + q, (\mathfrak{A}, \mathfrak{B})]$-cycle which, however, will be affected with a slight indetermination. To this end, we start with B^q and choose an auxiliary construction (sect. 2), which is always possible by sect. 3. Then we put

$$A^p B^q = a_i B^{p+q}(\sigma_i^p).$$

From (2.2) we have

$$F^* a_i B^{p+q}(\sigma_i^p) = a_i F^* B^{p+q}(\sigma_i^p) = \eta_{ji}^p a_i B^{p+q+1}(\sigma_j^{p+1})$$

which is equal to zero, since $\eta_{ji}^p a_i = 0$, A^p being a dual p-cycle. Therefore, $F^*(A^p B^q) = 0$, i.e., the product $A^p B^q$ is indeed a dual $(p + q)$-cycle.

It can easily be seen that the product $A^p B^q$ is *not* uniquely determined, depending really on the choice of the auxiliary construction. But the *homology class* of the product $A^p B^q$ *is* determined without ambiguity, i.e. any two values $(A^p B^q)_1$ and $(A^p B^q)_2$ are connected by the homology

$$(A^p B^q)_1 \sim (A^p B^q)_2.$$

This fact is an easy consequence of the following statement: If $B^q = 0$, then $A^p B^q \sim 0$ for any choice of the auxiliary construction. We proceed to the proof of that statement. If $B^q = 0$, we saw in sect. 4 that there exist chains $C^{p+q-1}(\sigma_i^p)$ such that (4.1) and (4.2) hold true. If $p = 0$, it follows from (4.1) that

$$A^0 B^q = a_i B^q(\sigma_i^0) = \eta_{ji}^0 a_i C^q(\sigma_i^0) = 0,$$

because $\eta_{ji}^0 a_i = 0$. If $p > 0$, it follows from (4.2) that

$$F^* a_i C^{p+q-1}(\sigma_i^p) = a_i F^* C^{p+q-1}(\sigma_i^p) = a_i B^{p+q}(\sigma_i^p) - \eta_{ji}^p a_i C^{p+q}(\sigma_j^{p+1})$$
$$= a_i B^{p+q}(\sigma_i^p) = A^p B^q$$

because $\eta_{ji}^p a_i = 0$. Therefore $A^p B^q = F^* a_i C^{p+q-1}(\sigma_i^p) \sim 0$.

The homology class of the product $A^p B^q$ is uniquely determined by the homology classes of A^p and B^q. This is an easy consequence of the following statement: If either $A^p \sim 0$ or $B^q \sim 0$, then $A^p B^q \sim 0$. Let us first suppose that $A^p \sim 0$. If $p = 0$, then $A^p = 0$, which implies $A^p B^q = 0$. If $p > 0$, then there exists a $(p - 1, \mathfrak{A})$-chain $\alpha_j \sigma_j^{p-1}$ such that $A^p = a_i \sigma_i^p = F^*(\alpha_j \sigma_j^{p-1})$, i.e. $a_i = \eta_{i\,j}^{p-1} \alpha_j$. According to (2.2), we have

$$F^* \alpha_j B^{p+q-1}(\sigma_j^{p-1}) = \alpha_j F^* B^{p+q-1}(\sigma_j^{p-1}) = \eta_{i\,j}^{p-1} \alpha_j B^{p+q}(\sigma_i^p) = a_i B^{p+q}(\sigma_i^p) = A^p B^q$$

so that $A^p B^q = 0$. Now we suppose that $B^q \sim 0$. If $q = 0$, then $B^q = 0$, which we know to imply $A^p B^q \sim 0$. If $q > 0$, then there exists a $(q - 1, \mathfrak{B})$-chain H^{q-1} such that $B^q = F^* H^{q-1}$. One finds easily $(q - 1, \mathfrak{B})$-chains $H^{q-1}(\sigma_i^0)$ such that (1) σ_i^0 is a vertex of every $(q - 1)$-simplex appearing in $H^{q-1}(\sigma_i^0)$,

(2) $H^{q-1} = \sum_i H^{q-1}(\sigma_i^0)$. If we put $B^q(\sigma_i^0) = F^*H^{q-1}(\sigma_i^0)$ and $B^{p+q}(\sigma_i^p) = 0$ for $p > 0$, we evidently have an auxiliary construction in the sense of sect. 2. With this choice of auxiliary construction, we have $A^pB^q = 0$ if $p > 0$, and

$$A^0B^q = F^*a_iH^{q-1}(\sigma_i^0) \sim 0 \text{ if } p = 0.$$

7. Let there be given an ordering of the vertices of the complex K. Then we can use the particular auxiliary construction described in sect. 3, which leads to the following simple definition of the product A^pB^q. Given a $(p + q)$-simplex σ^{p+q}, we write it as

$$\sigma^{p+q} = (v_0, v_1, \cdots, v_p, \cdots, v_{p+q})$$

according to the given ordering of the vertices. Let a be the coefficient of the p-simplex $(v_0, v_1, \cdots, v_p)$ in the $(p, \mathfrak{A})$-chain A^p; let b be the coefficient of the q-simplex $(v_p, \cdots, v_{p+q})$ in the $(q, \mathfrak{B})$-chain B^q. Then ab is the coefficient of σ^{p+q} in the product A^pB^q.

This definition leads to a simple proof of the *commutative law*:

$$B^qA^p \sim (-1)^{pq}A^pB^q. \tag{7.1}$$

Here we suppose that, $\mathfrak{A}$ and $\mathfrak{B}$ being two groups, A^p is a dual $(p, \mathfrak{A})$-cycle and B^q is a dual $(q, \mathfrak{B})$-cycle. Furthermore, an $(\mathfrak{A}, \mathfrak{B})$-multiplication is given, and hence an inverse $(\mathfrak{B}, \mathfrak{A})$-multiplication also (sect. 5). The products A^pB^q and B^qA^p are formed according to the first and second of these multiplications, respectively. To prove (7.1), we fix the value of A^pB^q according to a given ordering of the vertices, and fix B^qA^p according to the *inverse* ordering of the vertices. Let a $(p + q)$-simplex

$$(v_0, v_1, \cdots, v_p, \cdots, v_{p+q})$$

be written in the original ordering of the vertices. Since

$$(v_p, \cdots, v_0) = (-1)^{\frac{1}{2}p(p+1)}(v_0, \cdots, v_p),$$

$$(v_{p+q}, \cdots, v_p) = (-1)^{\frac{1}{2}q(q+1)}(v_p, \cdots, v_{p+q}),$$

$$(v_{p+q}, \cdots, v_p, \cdots, v_0) = (-1)^{\frac{1}{2}(p+q)(p+q+1)}(v_0, \cdots, v_p, \cdots, v_{p+q}),$$

$$\tfrac{1}{2}(p + q)(p + q + 1) = \tfrac{1}{2}p(p + 1) + \tfrac{1}{2}q(q + 1) + pq,$$

it is readily seen that, with our particular choice of the auxiliary construction, we have $B^qA^p = (-1)^{pq}A^pB^q$. It seems difficult to prove the commutative law (7.1) directly from the general definition given in sect. 6.

The *distributive laws*

$$(A_1^p + A_2^p)B^q \sim A_1^pB^q + A_2^pB^q, \tag{7.2}$$

$$A^p(B_1^q + B_2^q) \sim A^pB_1^q + A^pB_2^q \tag{7.3}$$

are immediate consequences of either of the two definitions of the product.

Now suppose that three groups $\mathfrak{A}_1$, $\mathfrak{A}_2$ and $\mathfrak{A}_3$ are given. Let there be given an $(\mathfrak{A}_1, \mathfrak{A}_2)$-multiplication and an $(\mathfrak{A}_2, \mathfrak{A}_3)$-multiplication. Further, putting

$$(\mathfrak{A}_1, \mathfrak{A}_2) = \mathfrak{A}_{12}, \qquad (\mathfrak{A}_2, \mathfrak{A}_3) = \mathfrak{A}_{23},$$

let us suppose that there is given an $(\mathfrak{A}_{12}, \mathfrak{A}_3)$-multiplication and an $(\mathfrak{A}_1, \mathfrak{A}_{23})$-multiplication. Suppose, finally, that the associative law

$$a_1a_2 \cdot a_3 = a_1 \cdot a_2a_3$$

holds true for $a_1 \epsilon \mathfrak{A}_1$, $a_2 \epsilon \mathfrak{A}_2$, $a_3 \epsilon \mathfrak{A}_3$. Then we have, if $A_i^{p_i}$ $(i = 1, 2, 3)$ is a dual $(p_i, \mathfrak{A}_i)$-cycle, the *associative law*

$$A_1^{p_1}A_2^{p_2} \cdot A_3^{p_3} \sim A_1^{p_1} \cdot A_2^{p_2}A_3^{p_3}. \tag{7.4}$$

The proof based on a given ordering of the vertices is quite trivial. A proof based directly on our general definition of the product is not difficult, however.

It would be interesting to prove, using only definitions based on the ordering of the vertices, that the homology class of the product A^pB^q is independent of the choice of the ordering.[3]

8. Let there be given an $(\mathfrak{A}, \mathfrak{B})$-multiplication. If $A^p = a_i\sigma_i^p$ is a $(p, \mathfrak{A})$-chain and if $B^p = b_i\sigma_i^p$ is a $(p, \mathfrak{B})$-chain, let us put

$$\varphi(A^p, B^p) = a_ib_i \epsilon (\mathfrak{A}, \mathfrak{B}).$$

If A^{p+1} is a $(p + 1, \mathfrak{A})$-chain and if B^p is a $(p, \mathfrak{B})$-chain, it is readily seen that

$$\varphi(FA^{p+1}, B^p) = \varphi(A^{p+1}, F^*B^p); \tag{8.1}$$

similarly we have

$$\varphi(A^p, FB^{p+1}) = \varphi(F^*A^p, B^{p+1}) \tag{8.2}$$

for any $(p, \mathfrak{A})$-chain A^p and any $(p + 1, \mathfrak{B})$-chain B^{p+1}.

9 Let there be given an $(\mathfrak{A}, \mathfrak{B})$-multiplication. Let A^{p+q} be an *ordinary* $(p + q, \mathfrak{A})$-cycle. Let B^q be a *dual* $(q, \mathfrak{B})$-cycle. We shall define a product $A^{p+q}B^q$ (not quite uniquely determined), which will be an *ordinary* $[p, (\mathfrak{A}, \mathfrak{B})]$-cycle. We choose an auxiliary construction $B^{p+q}(\sigma_i^p)$ associated with B^q (sect. 2), and we put

$$A^{p+q}B^q = c_i\sigma_i^p,$$

where (see sect. 8)

$$c_i = (-1)^{pq}\varphi[A^{p+q}, B^{p+q}(\sigma_i^p)].$$

That $A^{p+q}B^q$ is an ordinary $[p, (\mathfrak{A}, \mathfrak{B})]$-cycle, is trivial if $p = 0$. If $p > 0$,

[3] Such a proof has now been given by J. W. Alexander; see his paper cited above.

it follows from (2.2) and (8.1) that, for any $(p-1)$-simplex σ_j^{p-1},

$$(-1)^{pq}\eta_{i\,j}^{p-1}c_i = \eta_{i\,j}^{p-1}\varphi[A^{p+q}, B^{p+q}(\sigma_i^p)] = \varphi[A^{p+q}, \eta_{i\,j}^{p-1}B^{p+q}(\sigma_i^p)]$$
$$= \varphi[A^{p+q}, F^*B^{p+q-1}(\sigma_j^{p-1})] = \varphi[FA^{p+q}, B^{p+q-1}(\sigma_j^{p-1})] = \varphi[0, B^{p+q-1}(\sigma_j^{p-1})] = 0,$$

i.e. $F(A^{p+q}B^q) = 0$.

Suppose that $B^q = 0$. If $p = 0$, it follows from (4.1) that

$$\varphi[A^q, B^q(\sigma_i^0)] = \eta_{j\,i}^0\varphi[A^q, C^q(\sigma_j^1)],$$

so that

$$A^qB^q = F(\gamma_j\sigma_j^1), \qquad \gamma_j = \varphi[A^q, C^q(\sigma_j^1)],$$

i.e. $A^qB^q \sim 0$. If $p > 0$, it follows from (4.2) that

$$\varphi[A^{p+q}, B^{p+q}(\sigma_i^p)] = \eta_{j\,i}^p\varphi[A^{p+q}, C^{p+q-1}(\sigma_j^{p+1})] + \varphi[A^{p+q}, F^*C^{p+q-1}(\sigma_i^p)].$$

But the last summand is zero, from (8.1), since $FA^{p+q} = 0$. Therefore

$$A^{p+q}B^q = F(\gamma_j\sigma_j^{p+1}), \qquad \gamma_j = (-1)^{pq}\varphi[A^{p+q}, C^{p+q-1}(\sigma_j^{p+1})],$$

i.e. again $A^{p+q}B^q \sim 0$.

It follows readily from the preceding proof that, in any case, the homology class of the $[p, (\mathfrak{A}, \mathfrak{B})]$-cycle $A^{p+q}B^q$ is independent of the choice of the auxiliary construction. As a matter of fact, this homology class is uniquely determined by the homology classes of the ordinary $(p+q, \mathfrak{A})$-cycle A^{p+q} and the dual $(q, \mathfrak{B})$-cycle B^q. It is sufficient to prove that $A^{p+q}B^q \sim 0$, if either $A^{p+q} \sim 0$ or $B^q \sim 0$. If $A^{p+q} \sim 0$, there exists a $(p+q+1, \mathfrak{A})$-chain H^{p+q+1} such that $A^{p+q} = FH^{p+q+1}$. It follows easily from (2.2) and (8.1) that

$$A^{p+q}B^q = F(\gamma_j\sigma_j^{p+1}) \sim 0, \gamma_j = (-1)^{pq}\varphi[H^{p+q+1}, B^{p+q+1}(\sigma_j^{p+1})].$$

If $B^q \sim 0$ and $q = 0$, we have $B^q = 0$, which we know to imply $A^{p+q}B^q \sim 0$. If $B^q \sim 0$ and $q > 0$, we choose the auxiliary construction as at the end of sect. 6: $B^q(\sigma_i^0) = F^*H^{q-1}(\sigma_i^0)$ and $B^{p+q}(\sigma_i^p) = 0$ for $p > 0$. If $p > 0$, we have then $A^{p+q}B^q = 0$. If $p = 0$, we have again $A^qB^q = 0$ from (8.1), since $FA^q = 0$.

If A^p is a *dual* $(p, \mathfrak{A})$-cycle and if B^{p+q} is an *ordinary* $[(p+q), \mathfrak{B}]$-cycle, we put

$$A^pB^{p+q} = c_i\sigma_i^q,$$

where

$$c_i = \varphi[A^{p+q}(\sigma_i^p), B^{p+q}],$$

the $(p+q, \mathfrak{A})$-chains $A^{p+q}(\sigma_i^q)(q = 0, 1, 2, \cdots)$ being the elements of an auxiliary construction associated with A^p. Again, the product is an ordinary $[q, (\mathfrak{A}, \mathfrak{B})]$-cycle and only its homology class is uniquely determined, this class being indeed given by the mere knowledge of the homology classes of the factors. If A^{p+q} is an ordinary $(p+q, \mathfrak{A})$-cycle and if B^q is a dual $(q, \mathfrak{B})$-cycle,

we have evidently

(9.1) $$A^{p+q}B^q \sim (-1)^{pq}B^qA^{p+q},$$

where the left-hand member is defined according to the given $(\mathfrak{A}, \mathfrak{B})$-multiplication and the right-hand member according to the inverse $(\mathfrak{B}, \mathfrak{A})$-multiplication.

10. Let there be given an ordering of the vertices of the complex K. The particular auxiliary construction described in sect. 3 leads to following simple definition of the product A^pB^{p+q} of a dual $(p, \mathfrak{A})$-cycle A^p and an ordinary $(p+q, \mathfrak{B})$-cycle B^{p+q}. Given a q-simplex σ^q, we write it as

$$\sigma^q = (v_0, v_1, \cdots, v_q)$$

according to the given ordering of the vertices, and consider all the $(p+q)$-simplices

$$\sigma_k^{p+q} = (v_0, v_1, \cdots, v_q, \cdots, v_{p+q})$$

having σ^q as their common face and such that, in the given ordering, v_q precedes any vertex of σ_k^{p+q} which is not a vertex of σ^q. For every such σ_k^{p+q} put

$$\sigma_k^p = (v_q, \cdots, v_{p+q}).$$

Let a_k be the coefficient of σ_k^p in A^p; let b_k be the coefficient of σ_k^{p+q} in B^{p+q}. Then the coefficient of σ^q in A^pB^{p+q} is equal to

$$\sum_k a_k b_k.$$

Now let us consider the product $A^{p+q}B^q$ of an ordinary $(p+q, \mathfrak{A})$-cycle A^{p+q} and a dual $(q, \mathfrak{B})$-cycle B^q. This time we use the auxiliary construction based on the *inverse* ordering of the vertices, but we describe the result in terms of the original ordering. Given a p-simplex σ^p, we write it as

$$\sigma^p = (v_q, \cdots, v_{p+q})$$

according to the given ordering of the vertices, and consider all the $(p+q)$-simplices

$$\sigma_k^{p+q} = (v_0, v_1, \cdots, v_q, \cdots, v_{p+q})$$

having σ^p as their common face and such that, in the given ordering, v_q follows any vertex of σ_k^{p+q} which is not a vertex of σ^p. For every such σ_k^{p+q}, put

$$\sigma_k^q = (v_0, \cdots, v_q).$$

Let a_k be the coefficient of σ_k^{p+q} in A^{p+q}; let b_k be the coefficient of σ_k^q in B^q. Then the coefficient of σ^p in $A^{p+q}B^q$ is equal to

$$\sum_k a_k b_k.$$

These definitions, in connection with that given at the beginning of sect. 7

(for the product of two dual cycles) lead to a simple proof of the *associative laws*:

$$A_1^{p_1+p_2+p_3} B_2^{p_2} \cdot B_3^{p_3} \sim A_1^{p_1+p_2+p_3} \cdot B_2^{p_2} B_3^{p_3}, \tag{10.1}$$

$$B_1^{p_1} A_2^{p_1+p_2+p_3} \cdot B_3^{p_3} \sim B_1^{p_1} \cdot A_2^{p_1+p_2+p_3} B_3^{p_3}, \tag{10.2}$$

$$B_1^{p_1} B_2^{p_2} \cdot A_3^{p_1+p_2+p_3} \sim B_1^{p_1} \cdot B_2^{p_2} A_3^{p_1+p_2+p_3}. \tag{10.3}$$

Here we suppose given three groups $\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{A}_3$, an $(\mathfrak{A}_1, \mathfrak{A}_2)$-multiplication, an $(\mathfrak{A}_2, \mathfrak{A}_3)$-multiplication, an $(\mathfrak{A}_{12}, \mathfrak{A}_3)$-multiplication with $\mathfrak{A}_{12} = (\mathfrak{A}_1, \mathfrak{A}_2)$ and an $(\mathfrak{A}_1, \mathfrak{A}_{23})$-multiplication with $\mathfrak{A}_{23} = (\mathfrak{A}_2, \mathfrak{A}_3)$. It is supposed that $a_1a_2 \cdot a_3 = a_1 \cdot a_2a_3$ for $a_i \in \mathfrak{A}_i$ $(i = 1, 2, 3)$. $A_i^{p_1+p_2+p_3}$ $(i = 1, 2, 3)$ is an ordinary $(p_1 + p_2 + p_3, \mathfrak{A}_i)$-cycle and $B_i^{p_i}$ $(i = 1, 2, 3)$ is a dual $(p_i, \mathfrak{A}_i)$-cycle. Of course, any of the three formulas (10.1), (10.2) and (10.3) implies the others using (7.1) and (9.1). We omit writing explicitly the trivial *distributive laws*.

11. In the remaining part of this paper the coefficients of all chains are taken from the additive group of all integer numbers. Moreover, we suppose that $K = M_n$ is an orientable simple n-circuit, i.e. that the following four conditions are satisfied. *First*, each simplex of M_n is either an n-simplex or a face of an n-simplex. *Second*, each $(n - 1)$-simplex of M_n is a common face of precisely two n-simplices of M_n. *Third*, any two n-simplices of M_n may be connected by a sequence of n-simplices of M_n such that any two consecutive n-simplices of the sequence have a common $(n - 1)$-face. *Fourth*, the n-simplices σ_i^n of M_n can be given such orientations that their sum $\Gamma^n = \sum_i \sigma_i^n$ is an ordinary n-cycle. (We always suppose the orientation of the n-simplices chosen in this manner.)

If σ_i^p is any p-simplex of M_n, we denote by Lk. $[\sigma_i^p]$ its *link*, i.e. the subcomplex of M_n composed of all the simplices τ of M_n having no common vertex with σ_i^p but having the property that there exists a simplex of M_n having both τ and σ_i^p among its faces.

If $0 \leqq p \leqq n$, we say that M_n is *p-regular* if the following two conditions are satisfied. *First* (requiring nothing if $p = n$ or $p = n - 1$), the link Lk. $[\sigma_i^p]$ on any p-simplex of M_n is an orientable simple $(n - p - 1)$-circuit. Second (requiring nothing if $p = 0$), for each k such that $0 \leqq k \leqq p - 1$, any dual $(n - p - 1)$-cycle of any Lk. $[\sigma_i^k]$ is homologous to zero in Lk. $[\sigma_i^k]$. It is easily seen that the orientable combinatorial n-manifolds are identical with orientable simple n-circuits, which are p-regular for any $0 \leqq p \leqq n$.

12. For $0 \leqq p \leqq n$, we denote by $\mathfrak{B}_p$ the group of all the homology classes of ordinary p-cycles of M_n and by $\bar{\mathfrak{B}}_p$ the group of all the homology classes of dual p-cycles of M_n.

Given any dual $(n - p)$-cycle B^{n-p} of M_n $(0 \leqq p \leqq n)$, we put

$$\psi_p(B^{n-p}) = \Gamma^n \cdot B^{n-p},$$

where $\Gamma^n = \sum_i \sigma_i^n$. Evidently, ψ_p is a homomorphic mapping of the group $\bar{\mathfrak{B}}_{n-p}$ on a subgroup $\psi_p(\bar{\mathfrak{B}}_{n-p})$ of the group $\mathfrak{B}_p$.

13. *If M_n is p-regular, then the mapping ψ_p is* $1 - 1$, so that the group $\bar{\mathfrak{B}}_{n-p}$ is isomorphic with a subgroup [i.e. $\psi_p(\bar{\mathfrak{B}}_{n-p})$] of the group $\mathfrak{B}_p$.

It is sufficient to prove that $\Gamma^n B^{n-p} \sim 0$ implies $B^{n-p} \sim 0$.

Let $B^{n-p+k}(\sigma_i^k)$ be the elements of a given auxiliary construction associated with the dual $(n - p)$-cycle B^{n-p}. Since $\Gamma^n \cdot B^{n-p} \sim 0$, there exists a $(p + 1)$-chain $c_j\sigma_j^{p+1}$ such that $\Gamma^n \cdot B^{n-p} = (-1)^{p(n-p)}F(c_j\sigma_j^{p+1})$, i.e.

$$\varphi[\Gamma^n, B^n(\sigma_i^p)] = \eta_{j\,i}^{p}c_j \,.$$

For any σ_j^{p+1}, let us choose an n-simplex τ^n such that σ_j^{p+1} is a face of τ^n, and put $H^n(\sigma_j^{p+1}) = c_j\tau^n$. Since $\Gamma^n = \sum_i \sigma_i^n$, we have $\varphi[\Gamma_j^n H^n(\sigma_j^{p+1})] = c_j$ and, therefore,

$$\varphi[\Gamma^n, B_0^n(\sigma_i^p)] = 0 \,, \tag{13.1}$$

where

$$B_0^n(\sigma_i^p) = B^n(\sigma_i^p) - \eta_{j\,i}^{p}H^n(\sigma_j^{p+1}) \,. \tag{13.2}$$

Evidently σ_i^p is a face of each n-simplex appearing in the n-chain $B_0^n(\sigma_i^p)$. Therefore there exists in the link Lk. $[\sigma_i^p]$ an $(n - p - 1)$-chain $C^{n-p-1}(\sigma_i^p)$ such that the n-chain $B_0^n(\sigma_i^p)$ can be obtained from the $(n - p - 1)$-chain C^{n-p-1} by replacing each $(n - p - 1)$-simplex

$$(v_{p+1}, \cdots, v_n)$$

by the n-simplex

$$(v_0, \cdots, v_p, v_{p+1}, \cdots, v_n)$$

where

$$(v_0, \cdots, v_p) = \sigma_i^p \,. \tag{13.3}$$

Since the complex Lk. $[\sigma_i^p]$ contains no $(n - p)$-simplex, the $(n - p - 1)$-chain $C^{n-p-1}(\sigma_i^p)$ of the complex Lk. $[\sigma_i^p]$ must be a dual $(n - p - 1)$-cycle. Moreover, the equation (13.1) signifies that the sum of the coefficients of $C^{n-p-1}(\sigma_i^p)$ is equal to zero. Since M_n is p-regular, Lk. $[\sigma_i^p]$ is an orientable simple $(n - p - 1)$-circuit, which implies readily the existence of an $(n - p - 2)$-chain $D^{n-p-2}(\sigma_i^p)$ in the complex Lk. $[\sigma_i^p]$ such that

$$F^*D^{n-p-2}(\sigma_i^p) = (-1)^{p+1}C^{n-p-1}(\sigma_i^p) \,. \tag{13.4}$$

Let $H^{n-1}(\sigma_i^p)$ signify the $(n - 1)$-chain which arises from the $(n - p - 2)$-chain $D^{n-p-2}(\sigma_i^p)$ by replacing each $(n - p - 2)$-simplex

$$(v_{p+1}, \cdots, v_{n-1})$$

by the $(n-1)$-simplex

$$(v_0, \cdots, v_p, v_{p+1}, \cdots, v_{n-1}),$$

supposing the validity of (13.3). Then (13.4) implies that

$$F^*H^{n-1}(\sigma_i^p) = B_0^n(\sigma_i^p). \tag{13.5}$$

Moreover, σ_i^p is a face of every $(n-1)$-simplex appearing in the $(n-1)$-chain $H^{n-1}(\sigma_i^p)$.

Now, let us put

$$B_{p-1}^n(\sigma_i^p) = 0,$$
$$B_{p-1}^{n-1}(\sigma_j^{p-1}) = B^{n-1}(\sigma_j^{p-1}) - \eta_{i\,j}^{p-1} H^{n-1}(\sigma_i^p)$$

and

$$B_{p-1}^{n-p+k}(\sigma_i^k) = B^{n-p+k}(\sigma_i^k) \quad \text{for} \quad p-1 \neq k \neq p.$$

From (13.2) and (13.5) it is easily seen that the chains $B_{p-1}^{n-p+k}(\sigma_i^k)$ form an auxiliary construction associated with B^{n-p}.

Now let us suppose that (as we have found to be possible in the case $r = p - 1$) we have found chains $B_r^{n-p+k}(\sigma_i^k)(1 \leqq r \leqq p-1)$ forming an auxiliary construction associated with B^{n-p} and such that $B_r^{n-p+r+1}(\sigma_i^{r+1}) = 0$. By the definition of an auxiliary construction, we have

$$F^*B_r^{n-p+r}(\sigma_i^r) = 0 \tag{13.6}$$

for each σ_i^r. Since σ_i^r is a face of each $(n-p+r)$-simplex appearing in $B_r^{n-p+r}(\sigma_i^r)$, there exists in the link Lk. $[\sigma_i^r]$ an $(n-p-1)$-chain $C^{n-p-1}(\sigma_i^r)$ such that the $(n-p+r)$-chain $B_r^{n-p+r}(\sigma_i^r)$ can be obtained from the $(n-p-1)$-chain $C^{n-p-1}(\sigma_i^r)$ by replacing each $(n-p-1)$-simplex

$$(v_{r+1}, \cdots, v_{n-p+r})$$

by the $(n-p+r)$-simplex

$$(v_0, \cdots, v_r, v_{r+1}, \cdots, v_{n-p+r}),$$

where

$$(v_0, \cdots, v_r) = \sigma_i^r. \tag{13.7}$$

Now the equation (13.6) signifies that $C^{n-p-1}(\sigma_i^r)$ is a dual $(n-p-1)$-cycle of the complex Lk. $[\sigma_i^r]$. Since M_n is p-regular, it follows that there exists an $(n-p-2)$-chain $D^{n-p-2}(\sigma_i^r)$ of the complex Lk. $[\sigma_i^r]$ such that

$$F^*D^{n-p-2}(\sigma_i^r) = (-1)^{r+1}C^{n-p-1}(\sigma_i^r). \tag{13.8}$$

Let $H^{n-p+r-1}(\sigma_i^r)$ denote the $(n-p+r-1)$-chain which arises from the $(n-p-2)$-chain $D^{n-p-2}(\sigma_i^r)$ by replacing each $(n-p-2)$-simplex

$$(v_{r+1}, \cdots, v_{n-p+r-1})$$

by the $(n - p + r - 1)$-simplex

$$(v_0, \cdots, v_r, v_{r+1}, \cdots, v_{n-p+r-1}) .$$

supposing the validity of (13.7). Then (13.8) implies that

$$F^*H^{n-p+r-1}(\sigma_i^r) = B_r^{n-p+r}(\sigma_i^r) . \tag{13.9}$$

Now, let us put

$$\begin{gathered} B_{r-1}^{n-p+r}(\sigma_i^r) = 0 , \\ B_{r-1}^{n-p+r-1}(\sigma_j^{r-1}) = B_r^{n-p+r-1}(\sigma_j^{r-1}) - \eta_{i\,j}^{r-1} H^{n-p+r-1}(\sigma_i^r) \end{gathered} \tag{13.10}$$

and

$$B_{r-1}^{n-p+k}(\sigma_i^k) = B_r^{n-p+k}(\sigma_i^k) \quad \text{for} \quad r - 1 \neq k \neq r .$$

It follows readily from (13.9) that the chains $B_{r-1}^{n-p+k}(\sigma_i^k)$ form an auxiliary construction associated with B^{n-p} and such that (13.10) holds true.

Applying the preceding argument successively for $r = p - 1, p - 2, \cdots, 2, 1$, we obtain an auxiliary construction $B_0^{n-p+k}(\sigma_i^k)$ associated with B^{n-p} and such that $B_0^{n-p+1}(\sigma_i^1) = 0$. Applying the same argument again in the case $r = 0$, we have (13.9), written now as

$$F^*H^{n-p-1}(\sigma_i^0) = B_0^{n-p}(\sigma_i^0) .$$

But since $B_0^{n-p}(\sigma_i^0)$ are elements of an auxiliary construction associated with B^{n-p}, we have $B^{n-p} = \sum_i B_0^{n-p}(\sigma_i^0) = F^* \sum_i H^{n-p-1}(\sigma_i^0)$, whence $B^{n-p} \sim 0$.

14. *If M_n is $(p - 1)$-regular,*[4] *then the group $\psi_p(\bar{\mathfrak{B}}_{n-p})$ is the whole group $\mathfrak{B}_p$*, so that the group $\mathfrak{B}_p$ is a homomorphic image of the group $\bar{\mathfrak{B}}_{n-p}$. Comparing this with the result of the preceding section we see that, *if M_n is both $(p - 1)$-regular and p-regular, the groups $\mathfrak{B}_p$ and $\bar{\mathfrak{B}}_{n-p}$ are isomorphic.*

Let $C^p = c_i\sigma_i^p$ be an ordinary p-cycle of M_n, so that $\eta_{i\,j}^{p-1}c_i = 0$. We shall find a dual $(n - p)$-cycle B^{n-p} and an auxiliary construction $B^{n-p+k}(\sigma_i^k)$ associated with it such that $\Gamma^n \cdot B^{n-p} = C^p$, i.e.

$$\varphi[\Gamma^n, B^n(\sigma_i^p)] = c_i . \tag{14.1}$$

The construction of n-chains $B^n(\sigma_i^p)$ satisfying (14.1) is quite evident; it is sufficient to choose for each σ_i^p an n-simplex τ^n having σ_i^p among its faces and to put $B^n(\sigma_i^p) = c_i\tau^n$. Since $\eta_{i\,j}^{p-1}c_i = 0$, we have for each σ_j^{p-1}

$$\varphi[\Gamma^n, \eta_{i\,j}^{p-1}B^n(\sigma_i^p)] = 0 . \tag{14.2}$$

Since σ_j^{p-1} is a face of every n-simplex appearing in $\eta_{i\,j}^{p-1}B^n(\sigma_i^p)$ and since the $(p-1)$-regularity of M_n implies that the link Lk. $[\sigma_j^{p-1}]$ is an orientable simple $(n - p)$-circuit, we can start with (14.2) and repeat the same argument which, in the preceding section and starting with (13.1), led us to (13.5). We thus

[4] Any M_n is supposed to be (-1)-regular.

obtain, for every σ_j^{p-1}, an $(n-1)$-chain $B^{n-1}(\sigma_j^{p-1})$ such that σ_j^{p-1} is a face of each simplex appearing in $B^{n-1}(\sigma_j^{p-1})$ and such that

$$F^*B^{n-1}(\sigma_j^{p-1}) = \eta_{i\,j}^{p-1}B^n(\sigma_i^p)\ .$$

More generally, let us suppose that, for a given $r(1 \leqq r \leqq p-1)$, we have succeeded in attaching to every σ_i^k $(r \leqq k \leqq p)$ an $(n-p+k)$-chain $B^{n-p+k}(\sigma_i^k)$ having the two following properties. *First*, σ_i^k is a face of each $(n-p+k)$-simplex appearing in $B^{n-p+k}(\sigma_i^k)$. *Second*, we have for $r \leqq k \leqq p-1$

$$F^*B^{n-p+k}(\sigma_i^k) = \eta_{j\,i}^k B^{n-p+k+1}(\sigma_j^{k+1})\ . \tag{14.3}$$

It follows that

$$F^*\eta_{i\,j}^{r-1}B^{n-p+r}(\sigma_i^r) = 0\ . \tag{14.4}$$

Since σ_j^{r-1} is a face of every $(n-p+r)$-simplex appearing in $\eta_{i\,j}^{r-1}B^{n-p+r}(\sigma_i^r)$ and since the $(p-1)$-regularity of M_n implies that every dual $(n-p-1)$-cycle of the complex Lk. $[\sigma_j^{r-1}]$ is homologous to zero in Lk. $[\sigma_j^{r-1}]$, we can start with (14.4) and repeat the same argument which, in the preceding section and starting with (13.6), led us to (13.9). We obtain thus, for every σ_j^{r-1}, an $(n-p+r-1)$-chain $B^{n-p+r-1}(\sigma_j^{r-1})$ such that σ_j^{r-1} is a face of each simplex appearing in $B^{n-p+r-1}(\sigma_j^{r-1})$ and such that (14.3) holds true for $k = r-1$.

Starting with the chains $B^n(\sigma_i^p)$ and $B^{n-1}(\sigma_i^{p-1})$ already found, and applying the preceding argument successively for $r = p-1, p-2, \cdots, 2, 1$, we find chains $B^{n-p+k}(\sigma_i^k)$ $(0 \leqq k \leqq p)$ such that σ_i^k is a face of each simplex appearing in $B^{n-p+k}(\sigma_i^k)$ and such that (14.3) holds true for $0 \leqq k \leqq p-1$. In particular, for $k = 0$, (14.3) says that

$$F^*B^{n-p}(\sigma_i^0) = \eta_{j\,i}^0 B^{n-p+1}(\sigma_j^1)\ .$$

Since $\sum_i \eta_{j\,i}^0 = 0$ for every σ_j^1, we have $F^* \sum_i B^{n-p}(\sigma_i^0) = 0$, i.e.

$$B^{n-p} = \sum_i B^{n-p}(\sigma_i^0)$$

is a dual $(n-p)$-cycle. Of course our chains $B^{n-p+k}(\sigma_i^k)$ form an auxiliary construction associated with B^{n-p} and we have $\Gamma^n \cdot B^{n-p} = C^p$.

15. Let $0 \leqq p \leqq n$, $0 \leqq q \leqq n$. Suppose that M_n is r-regular both for $r = p$ and for $r = q$. Let C^p be an ordinary p-cycle belonging to the family $\psi_p(\bar{\mathfrak{B}}_{n-p})$; let D^q be an ordinary q-cycle belonging to the family $\psi_q(\bar{\mathfrak{B}}_{n-q})$; if M_n is r-regular also for $r = p-1$ and $r = q-1$, we know (sect. 14) that the cycles C^p and D^q are unrestricted.

We shall define the *intersection* of C^p and D^q and we shall designate it by $C^p \times D^q$. In the case $p + q < n$ we simply put

$$C^p \times D^q = 0\ .$$

In the case $p + q \geqq n$, we shall define $C^p \times D^q$ as an ordinary $(p+q-n)$-cycle, but only its homology class will be uniquely determined.

Since C^p belongs to $\psi_p(\mathfrak{B}_{n-p})$, there exists a dual $(n-p)$-cycle A^{n-p} such that

$$\Gamma^n A^{n-p} \sim C^p \,. \tag{15.1}$$

Since D^q belongs to $\psi_q(\mathfrak{B}_{n-q})$, there exists a dual $(n-q)$-cycle B^{n-q} such that

$$\Gamma^n B^{n-q} \sim D^q \,. \tag{15.2}$$

We know (see sect. 13) that the homology classes of A^{n-p} and B^{n-q} are uniquely defined.

This being done, we put

$$C^p \times D^q \sim \Gamma^n \cdot A^{n-p} B^{n-q} \,. \tag{15.3}$$

It follows from (10.1) and (15.1) that

$$C^p \times D^q \sim C^p B^{n-q} \,. \tag{15.4}$$

The *distributive laws*

$$\begin{aligned} (C_1^p + C_2^p) \times D^q &\sim (C_1^p \times D^q) + (C_2^p \times D^q) \,, \\ C^p \times (D_1^q + D_2^q) &\sim (C^p \times D_1^q) + (C^p \times D_2^q) \end{aligned} \tag{15.5}$$

are evident. The *commutative law*

$$D^q \times C^p \sim (-1)^{(n-p)(n-q)} C^p \times D^q \tag{15.6}$$

follows from (7.1) and (15.3). If M_n is also s-regular and if E^s is an ordinary s-cycle belonging to the family $\psi_s(\mathfrak{B}_{n-s})$, we see from (7.4), (10.1) and (15.3) the validity of the *associative law*

$$(C^p \times D^q) \times E^s \sim C^p \times (D^q \times E^s) \,. \tag{15.7}$$

16. Let M_n be an orientable combinatorial n-manifold and let M_n' be its barycentrical subdivision. It is well known that M_n' is also an orientable combinatorial n-manifold. We shall show that, on the manifold M_n', our definition of intersection of ordinary cycles is equivalent to the classical definition.

Let $\sigma_i^p (0 \leqq p \leqq n)$ denote the simplices of M_n. We choose the orientation of the n-simplices σ_i^n in such manner that $\gamma^n = \sum_i \sigma_i^n$ is an ordinary n-cycle on M_n; we choose arbitrarily the orientation of the p-simplices $\sigma_i^p (1 \leqq p \leqq n-1)$ and, as usual, we denote by $\eta_{i\,j}^{p}$ the incidence coefficient of σ_i^{p+1} and $\sigma_j^p (0 \leqq p \leqq n-1)$.

Now let us recall the definition of the complex M_n'. The vertices of M_n' are identical with the simplices $\sigma_i^p (0 \leqq p \leqq n)$ of M_n. The vertices $\sigma_{i_0}^{p_0}$, $\sigma_{i_1}^{p_1}, \cdots, \sigma_{i_r}^{p_r}$ of M_n', where $p_0 \leqq p_1 \leqq \cdots \leqq p_r$, form an r-simplex of M_n' if and only if (1) $p_0 < p_1 < \cdots < p_r$, (2) $\sigma_{i_s}^{p_s}$ is a face of $\sigma_{i_{s+1}}^{p_{s+1}}$ for $0 \leqq s \leqq r-1$.

Put

$$\Gamma^n = \sum \eta^{0}_{i_1 i_0} \eta^{1}_{i_2 i_1} \cdots \eta^{n-1}_{i_n i_{n-1}} (\sigma^0_{i_0}, \sigma^1_{i_1}, \cdots, \sigma^n_{i_n})$$

the summation running over all the n-simplices of M'_n. It is well known that Γ^n is an ordinary n-cycle of M'_n (usually called the barycentrical subdivision of γ^n).

The classical intersection of two ordinary cycles on M'_n is obtained by choosing each factor in a particular way in its homology class, which we must describe in detail.

Let $H^p = a_i \sigma^p_i$ be an ordinary p-cycle of M_n. Put

$$C^p = \sum \eta^{0}_{i_1 i_0} \eta^{1}_{i_2 i_1} \cdots \eta^{p-1}_{i_p i_{p-1}} a_{i_p} (\sigma^0_{i_0}, \sigma^1_{i_1}, \cdots, \sigma^p_{i_p}),$$

the summation running over all the p-simplices of M'_n having the indicated form $(\sigma^0_{i_0}, \sigma^1_{i_1}, \cdots, \sigma^p_{i_p})$. Let $K^{n-q} = b_i \sigma^{n-q}_i$ be a dual $(n - q)$-cycle of M_n. Put

$$D^q = \sum \eta^{n-q}_{i_{n-q+1} i_{n-q}} \cdots \eta^{n-1}_{i_n i_{n-1}} b_{i_{n-q}} (\sigma^{n-q}_{i_{n-q}}, \sigma^{n-q+1}_{i_{n-q+1}}, \cdots, \sigma^n_{i_n}),$$

the summation running over all the q-simplices of M'_n having the indicated form $(\sigma^{n-q}_{i_{n-q}}, \sigma^{n-q+1}_{i_{n-q+1}}, \cdots, \sigma^n_{i_n})$.

In the classical theory of combinatorial manifolds it is shown that C^p is an ordinary p-cycle on M'_n, that D^q is an ordinary q-cycle on M'_n, and that we may choose the ordinary p-cycle H^p on M_n and the dual $(n - q)$-cycle K^{n-q} on M_n in such a manner that C^p and D^q are homologous to arbitrarily given ordinary p-cycle and q-cycle on M'_n. The classical intersection of C^p and D^q is zero if $p + q < n$; in the case $p + q \geqq n$, it is equal to

$$C^p \times D^q = \sum \eta^{n-q}_{i_{n-q+1} i_{n-q}} \cdots \eta^{p-1}_{i_p i_{p-1}} a_{i_p} b_{i_{n-q}} (\sigma^{n-q}_{i_{n-q}}, \cdots, \sigma^p_{i_p}), \tag{16.1}$$

the summation running over all the $(p + q - n)$-simplices of M'_n having the indicated form $(\sigma^{n-q}_{i_{n-q}}, \cdots, \sigma^p_{i_p})$.

The case $p + q < n$ being trivial, we have to show that, if $p + q \geqq n$, (16.1) holds true according to our definition of intersection.

We now choose an ordering ω of the vertices of M_n and define an $(n - q)$-chain B^{n-q} on M'_n as follows. Let

$$\tau^{n-q} = (\sigma^{h_0}_{i_0}, \sigma^{h_1}_{i_1}, \cdots, \sigma^{h_{n-q}}_{i_{n-q}})$$

be an $(n - q)$-simplex of M'_n. Let v_λ be the first vertex of the h_λ-simplex $\sigma^{h_\lambda}_{i_\lambda}$ $(0 \leqq \lambda \leqq n - q)$, relatively to the ordering ω. If the v_λ's $(0 \leqq \lambda \leqq n - q)$ are not all different from each other, then the coefficient of τ^{n-q} in B^{n-q} will be zero. In the other case,

$$(v_0, v_1, \cdots, v_{n-q}) \tag{16.2}$$

is an $(n - q)$-simplex of M_n and the coefficient of τ^{n-q} in B^{n-q} will be equal to the coefficient of (16.2) in K^{n-q}. It is not difficult to verify that B^{n-q} is a dual cycle on M'_n.

Now we order the set of all the vertices of M_n' in such a manner that σ_i^h precedes σ_j^k, whenever $h < k$; this can be done in many ways. We form the product $\Gamma^n B^{n-q}$ in the manner explained in sect. 10, using our ordering of the vertices of M_n'. We easily verify that

$$\Gamma^n B^{n-q} = D^q,$$

so that

$$C^p \times D^q \sim C^p B^{n-q}$$

from (15.2) and (15.3). Now if we form the product $C^p B^{n-q}$ again in the manner explained in sect. 10, using the same ordering of the vertices of M_n', we easily verify that (16.1) holds true.

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ON THE CONNECTIVITY RING OF AN ABSTRACT SPACE

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I. Introduction

1. The k-dimensional connectivity numbers of an ordinary, smooth n-space may either be obtained *geometrically* by calculating the maximal number of k-cycles of the space that are independent modulo the bounding cycles or *analytically* by calculating the maximal number of exact, k-dimensional symbolic differential forms

$$w = \sum A_{i_1 \cdots i_k} dx_{i_1} \cdots dx_{i_k}$$

(integrands of multiple integrals) that are independent, in the large, modulo the derived forms. The geometrical method of approach has been extended to compact metric spaces by Vietoris[1] and to still more general spaces by Čech.[2] Moreover, this branch of the theory has been very greatly perfected by the introduction of Pontrjagin's cycles with real coefficients reduced modulo 1.

Now, if we use Pontrjagin's cycles, the k^{th} connectivity group of a compact, metric space becomes a compact, metric group. Moreover, by a theorem of Pontrjagin,[3] every such group may be identified with the character group of a countable, discrete group. This immediately suggests the advisability of regarding the discrete group, rather than its equivalent (though more complicated) metric character group, as the k^{th} invariant of the space, and of looking for a revised theoretical treatment leading simply and directly to this group. We give such a treatment below, based on a suitable combinatory adaptation of the second, or analytic, method of approach.

One decided advantage of taking the discrete groups rather than their metric character groups as the fundamental connectivity groups of the space is that we can then define the *product*[4] (as distinguished from the sum) of two elements of the same or of different groups. The combined groups of all dimensionalities (or, more precisely, their direct sum) will thus become a *connectivity ring*, as distinguished from a set of isolated connectivity groups.

2. A rough sketch of the present theory was given by the author in two

[1] L. Vietoris: "Über den höheren Zusammenhang kompakter Räume und eine Klasse von zusammenhangstreuen Abbildungen," *Math. Ann.* 97 (1927), pp. 454–472.

[2] E. Čech: "Théorie générale de l'homologie dans un espace quelconque," *Fund. Math.* 19 (1932), pp. 149–183.

[3] L. Pontrjagin: "The theory of topological commutative groups," these *Annals of Mathematics*, 35 (1934), pp. 361–388.

[4] The bracket product of §11.

notes which appeared in the *Proceedings of the National Academy*,[5] and in a paper read at the First International Topological Conference in Moscow.[6] Essentially the same theory was also developed, quite independently, by A. Kolmogoroff and likewise presented by him at the Moscow Congress. The definition of the product of two group elements (the analogue of the product of two symbolic differential forms) as originally given by Kolmogoroff and the present author has recently been modified by Čech,[7] so that the revised product is a p^{th} part of the original one, where p is a constant depending on the dimensionalities of the two factors. This modification is an essential improvement, as may be seen, for example, when the theory is applied to manifolds.[8] Nor is it in any way trivial, for in the theory as originally developed it is not clear, *a priori*, that the product of two elements has a p^{th} part. We have adopted Čech's modified definition of the product in this paper, although the change has necessitated a complete re-casting of all our proofs. The present paper deals only with the most fundamental definitions and theorems. Further developments of the theory will be given subsequently.

II. Symbolic Complexes and Spaces

3. We start with an arbitrary set of entities v_i called (*symbolic*) *vertices*. Any set of $n + 1$ distinct vertices, ($n = 0, 1, 2, \cdots$) will be called a (*symbolic*) *n-simplex*, or simplex of dimensionality n. Moreover, for the sake of uniformity, the null set (containing no vertices at all) will be called a simplex of dimensionality -1. The null set will be the only simplex of negative dimensionality.

A simplex S will be a *component* of a simplex T if every vertex of S is a vertex of T. If S is a component of T the vertices of T, but not of S, will determine a component S' of T called the *complement* of S with respect to T. Two simplexes will be *incident* if either is a component of the other. The null set will, of course, be a component of every simplex S and, therefore, incident to every S; a simplex S will be a component of itself and, therefore, incident to itself.

Now, let S be any simplex of dimensionality greater than zero, and let the vertices of S be $x_0, x_1, \cdots, x_n$, ($n \geqq 1$). Then the various possible permutations of the vertices x_i may be divided into two classes such that two permutations belong to the same or to different classes according as they differ by an even or by an odd number of inversions. We shall call a definite, though arbitrary, one of the two classes the positive class and the other the negative class

[5] "On the chains of a complex and their duals," and "On the ring of a compact metric space," *Proc. Nat. Acad.* 21 (1935), pp. 509–512.

[6] September 1935.

[7] "Multiplications on a complex" by E. Čech in the present number of these *Annals*, pp. 681–697. A similar modification was also suggested independently by H. Whitney in a letter to L. Zippin. The definition of the product announced by the author in the second of the two *Proceedings* notes referred to above is not the significant one, as was noticed by him while the note was in press. His revised definition was equivalent to Kolmogoroff's.

[8] Cf. Čech, loc. cit.

thereby *orienting* the simplex. The simplex can have either of two opposite orientations, corresponding to the two possible ways of naming the permutation classes. If s denotes an oriented simplex, then $-s$ will denote the same simplex with the opposite orientation. An oriented simplex will often be denoted by a positive permutation of its vertices, $s = x_0, x_1, \cdots, x_n$. We shall not orient the simplex of dimensionality -1 nor the simplexes of dimensionality 0.

4. A (*symbolic*) *complex* will be any set of simplexes such that there is at least one simplex in the set and such that every component of every simplex of the set is in the set. The set need not be finite nor even countable; it may contain simplexes of all dimensionalities. Every complex will contain the null set. A complex Π' will be called a *subcomplex* of a complex Π if every simplex of Π' is a simplex of Π.

5. A (*symbolic*) *space* will be any set of complexes Π_σ such that to each pair of complexes Π_{σ_1}, and Π_{σ_2} of the set there corresponds at least one complex Π_{σ_3} of the set which is a common subcomplex of both Π_{σ_1} and Π_{σ_2}. To illustrate the topological significance of this definition we shall show the connection between a symbolic space, as here defined, and an ordinary abstract space in the sense of classical point set theory. Consider an arbitrary point set Σ. A *covering* $[\sigma]$ of Σ will be any set of subsets σ of Σ such that each point of Σ belongs to at least one of the subsets σ. A covering $[\sigma_1]$ will be a *refinement* of a covering $[\sigma]$ if every element σ_1 of $[\sigma_1]$ is a subset of at least one element σ of $[\sigma]$. An abstract space will consist, essentially, of a point set Σ and of a set of coverings $[\sigma]$ of Σ such that to each pair of coverings $[\sigma_1]$ and $[\sigma_2]$ of the set there corresponds at least one covering $[\sigma_3]$ which is a common refinement of both $[\sigma_1]$ and $[\sigma_2]$. (We shall get specific types of spaces by making specific assumptions about the coverings.) Now, every abstract space determines a definite symbolic space such that: (a) the points of the abstract space are the underlying vertices of the symbolic space; (b) each covering $[\sigma]$ of the abstract space determines a complex Π_σ of the symbolic space consisting of all simplexes such that their vertices belong to the same element of the covering $[\sigma]$. Since two coverings $[\sigma_1]$ and $[\sigma_2]$ always have a common refinement, two complexes Π_{σ_1} and Π_{σ_2} will always have a common subcomplex Π_{σ_3}. Therefore, the complexes Π_σ will determine a symbolic space, as required.

A symbolic space will be more general than an abstract space, in the sense that not every symbolic space can be determined by an abstract space.

III. Skew Symmetric Functions on a Complex

6. Let R be an arbitrary ring (which need not necessarily be commutative.[9] An *m-function*, ($m = 1, 2, \cdots$), will be any skew-symmetrical function

[9] It can, perhaps, be shown that the only essentially independent invariants of a complex or space are the ones obtained by taking R to be the ring of all integers. This question should be further investigated.

$\varphi(x_0, \cdots, x_m)$ of $m + 1$ variable vertices x_i, such that the values of the function are numbers in the ring R. When the vertices x_i are all distinct they determine an oriented m-simplex $s = x_0, x_1, \cdots, x_m$. The value of the function φ corresponding to the vertices of s will be called the value of φ on s. Since the function φ is skew symmetrical, every even permutation of the vertices x_i leaves its value unaltered and every odd permutation merely changes its sign. The function φ will therefore have a unique value on s, and its value on the oppositely oriented simplex $-s$ will be the negative of its value on s. If two or more of the vertices x_i are identical the value of φ will, of course, be zero. A 0-*function* will simply be a function $\varphi(x)$ of a single variable vertex x. It will have a definite value on each 0-simplex. For the sake of uniformity, we shall regard the elements of the ring R as functions of dimensionality -1 defined on the null set (the simplex of dimensionality -1).

An m-function ($m \geqq -1$) will be said to *vanish* on a complex Π if it vanishes on every m-simplex of Π. Two m-functions will be said to be *identical* on Π if their difference vanishes on Π.

7. With every m-function φ there is associated an $(m + 1)$-function φ' determined by the following relation[10]

$$\varphi'(x_0, \cdots, x_{m+1}) = \sum_0^{m+1} (-1)^i \varphi(x_0, \cdots, \hat{x}_i, \cdots, x_{m+1}) . \tag{7:1}$$

We shall call the function φ' the *derivative* of the function φ. Its value on any oriented $(m + 1)$-simplex $s = x_0, \cdots, x_{m+1}$ will clearly be the sum of the values of φ on the m-components of s, with due regard to the proper orientation of the latter. The proof that the function φ' is skew symmetrical and, therefore, actually an $(m + 1)$-function follows, at once, from the obvious fact that when we permute any two consecutive vertices x_s and x_{s+1} we merely change the sign of the function.

8. THEOREM 1. *The second derivative φ'' of φ (i.e., the derivative of the derivative) always vanishes identically.*

For we have

$$\begin{aligned}\varphi''(x_0, \cdots, x_{m+2}) &= \sum_0^{m+2} (-1)^i \varphi'(x_0, \cdots, \hat{x}_i, \cdots, x_{m+2}) \\ &= \sum_{k<i} (-1)^{k+i} \varphi(x_0, \cdots, \hat{x}_k, \cdots, \hat{x}_i, \cdots, x_{m+2}) \\ &\quad + \sum_{k>i} (-1)^{k+i-1} \varphi(x_0, \cdots, \hat{x}_i, \cdots, \hat{x}_k, \cdots, x_{m+2}) .\end{aligned}$$

Moreover, the two sums in the last member cancel one another, since they only

[10] The symbol $\hat{x}_i$ will be used throughout to denote the absence of the variable x_i. Thus

$$\varphi(x_0, \cdots, \hat{x}_i, \cdots, x_{m+1}) = \varphi(x_0, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{m+1}).$$

differ in sign. (This may be seen by permuting the dummy indices k and i in either one of them.)

An m-function will be said to be *exact* (with reference to a complex Π) if its derivative vanishes on Π; it will be said to be *derived* (with reference to Π) if it is identical on Π with the derivative of some $(m - 1)$-function ψ.

COROLLARY. *Every derived function φ is exact.*

Because

$$\varphi = \psi' \qquad \text{(on } \Pi\text{)}$$

implies

$$\varphi' = \psi'' = 0 \qquad \text{(on } \Pi\text{)}.$$

9. We shall next define a law of composition according to which an m-function φ and an n-function ψ will combine to determine an $(m + n)$-function $[\varphi\psi]$, called the *bracket product* of φ and ψ. Let us arrange the underlying vertices v_i in a definite, though arbitrary, linear order α.[11] Then, if the variables $x_0, x_1, \cdots, x_{m+n}$ represent distinct vertices in normal order with respect to α, we shall assign to the product $[\varphi\psi]$ the value

$$(9:1) \qquad [\varphi\psi](x_0, \cdots, x_{m+n}) = \varphi(x_0, \cdots, x_m)\,\psi(x_m, \cdots, x_{m+n}),$$

where the expression on the right is the ordinary product of two elements of the ring R. Since the function $[\varphi\psi]$ is skew symmetric, its value will now be determined for arbitrary choices of the variables x_i. (It should be noticed that the functions φ and ψ in the right-hand member of (9:1) both involve the variable x_m.)

Since the right-hand member of (9:1) is an ordinary product in the ring R we can verify, at once, that bracket multiplication is distributive with respect to addition:

$$(9:2) \qquad [(\varphi_1 + \varphi_2)(\psi_1 + \psi_2)] = [\varphi_1\psi_1] + [\varphi_1\psi_2] + [\varphi_2\psi_1] + [\varphi_2\psi_2].$$

Of course, the bracket product is also associative though not, in general, commutative.

The function $[\varphi\psi]$ depends essentially on the linear ordering α of the vertices. When we wish to emphasize this dependence we shall write $[\varphi\psi]_\alpha$ in place of $[\varphi\psi]$.

10. THEOREM 2. *If φ is any m-function and ψ any n-function, then the deriva-*

[11] The idea of defining the bracket product $[\varphi\psi]$ with reference to a specific ordering of the vertices v_i is due to Čech and Whitney. In the original definitions of Kolmogoroff and the present author, the vertices were not ordered, and the product consisted of a set of terms corresponding to various permutations of the vertices. An extraneous numerical factor was thus introduced.

tive of the bracket product $[\varphi\psi]$ *satisfies the relation*

$$[\varphi\psi]' = [\varphi'\psi] + (-1)^m [\varphi\psi']. \tag{10:1}$$

For we may write

$$[\varphi\psi]'(x_0, \cdots, x_{m+n+1}) = \sum_{i=0}^{m+n+1} (-1)^i [\varphi\psi](x_0, \cdots, \hat{x}_i, \cdots, x_{m+n+1})$$

$$= \sum_0^m (-1)^i \varphi(x_0, \cdots, \hat{x}_i, \cdots, x_{m+1}) \psi(x_{m+1}, \cdots, x_{m+n+1})$$

$$+ \sum_{m+1}^{m+n+1} (-1)^i \varphi(x_0, \cdots, x_m) \psi(x_m, \cdots, \hat{x}_i, \cdots, x_{m+n+1}) .$$

Moreover, if we add the sum $(-1)^{m+1}\varphi(x_0, \cdots, x_m)\psi(x_{m+1}, \cdots, x_{m+n+1})$ to the first sum in the last expression, and substract it from the second, we obtain, at once, the desired relation (10:1).

COROLLARY 1. *The bracket product* $[\varphi\psi]$ *of two exact functions* φ *and* ψ *is always exact.*

For the functions φ' and ψ' vanish on Π. Therefore, by (10:1), the function $[\varphi\psi]'$ also vanishes on Π.

COROLLARY 2. *The bracket product* $[\varphi\psi]$ *of an exact function by a derived function or of a derived function by an exact function is a derived function.*

PROOF. Suppose, for instance, that φ is exact and ψ derived. Then there exists a function θ such that

$$\psi = \theta' \qquad \text{(on } \Pi) .$$

Moreover, we also have

$$\varphi' = 0 \qquad \text{(on } \Pi) ;$$

therefore, by (10:1)

$$[\varphi\theta]' = (-1)^m [\varphi\theta'] \qquad \text{(on } \Pi) ;$$

therefore, finally

$$[(-1)^m\varphi\theta]' = [\varphi\theta'] = [\varphi\psi] \qquad \text{(on } \Pi) .$$

The case where φ is derived and ψ exact is treated in essentially the same manner.

IV. THE CONNECTIVITY RING OF A COMPLEX

11. Let Π be an arbitrary complex. Then the set of all m-functions that are exact with respect to Π may be regarded as an abelian group G_E^m under the operation of addition. Moreover, the set of all m-functions that are derived with respect to Π may be regarded as a subgroup G_D^m of the group G_E^m, by the corollary to Theorem 1. We shall call the residue group $G^m = G_E^m \bmod G_D^m$ the m^{th} *connectivity group* of Π. Each element Φ of the group G^m will be a class

of exact functions, such that two functions belong to the same class if, and only if, their difference is a derived function.

The groups G^m ($m = 0, 1, \cdots$) are inter-related in the following manner. Let Φ be an arbitrary element of the m^{th} group G^m and Ψ an arbitrary element of the n^{th} group G^n. Then the elements Φ and Ψ together determine an element $[\Phi\Psi]$ of the group G^{m+n} called the *bracket product* of Φ and Ψ. The element $[\Phi\Psi]$ will be the class containing the function $[\varphi\psi]$, where φ and ψ are arbitrary functions of the classes Φ and Ψ respectively. It is easy to see that the class $[\Phi\Psi]$ is independent of the choice of the functions φ and ψ within their respective classes. For the class Φ is composed of exact functions of the type

$$\varphi = \varphi_0 + f,$$

where φ_0 is an arbitrary element of the class and f an arbitrary derived function. Similarly, the class Ψ is composed of exact functions of the type

$$\psi = \psi_0 + g$$

where ψ_0 is in the class and g a derived function. Now, by the distributive law of bracket multiplication, we have

$$(11:1) \qquad [\varphi\psi] = [\varphi_0\psi_0] + [\varphi_0 g] + [f\psi_0] + [fg].$$

Moreover, by Corollary 2 of Theorem 2, the last three terms on the right are derived functions. Therefore, the functions $[\varphi\psi]$ all belong to the same class as the functions $[\varphi_0\psi_0]$.

12. Theorem 3. *The bracket product* $[\Phi\Psi]$ *is independent of the ordering* α *of the vertices* v_i *and is, therefore, a function of the group elements* Φ *and* Ψ *alone.*

Proof. We shall first consider the case where the number of underlying vertices v_i is finite, since the general case involves a slight added complication. When the number of vertices is finite we can pass from any linear arrangement α to any other linear arrangement β by a finite sequence of simple steps, such that at each step we merely permute two consecutive vertices. The problem therefore reduces to showing that if the arrangement β differs from the arrangement α by a single permutation of two consecutive vertices v_r and v_{r+1} of α then the difference $[\varphi\psi]_\beta - [\varphi\psi]_\alpha$ is identical on Π with a derived function ζ'. We shall actually construct an $(m + n - 1)$-function ζ such that its derivative satisfies the relation

$$(12:1) \qquad \zeta' = [\varphi\psi]_\beta - [\varphi\psi]_\alpha \qquad (\text{on } \Pi).$$

Since the function ζ must be skew symmetrical (when it involves more than one variable) it will be sufficient to define its value for the case where the vertices $x_0, \cdots, x_{m+n-1}$ are all distinct and in normal order with respect to α.

Under these circumstances, we shall write

$$(12:2)\ \zeta(x_0, \cdots, x_{m+n-1}) = \begin{cases} (-1)^{m+1}\varphi(x_0, \cdots, x_{m-1}, x_m)\psi(x_{m-1}, x_m, \cdots, x_{m+n-1}) \\ \text{or} \\ 0, \end{cases}$$

where the upper determination is to hold when the two particular variables x_{m-1} and x_m have the values v_r and v_{r+1} respectively, while the lower one is to hold in all other cases. To complete the argument, we merely have to verify that the derivative ζ' of ζ,

$$(12:3)\qquad \zeta'(x_0, \cdots, x_{m+n}) = \sum_0^{m+n} (-1)^i \zeta(x_0, \cdots, \hat{x}_i, \cdots, x_{m+n}),$$

satisfies Relation (12:1). Here, again, we may, of course, assume that the vertices x_i are distinct and in normal order with respect to α.

For convenience, we shall consider four separate cases which will exhaust all the various possibilities.

Case 1. The two vertices v_r and v_{r+1} are not both present among the vertices x_i.

Then, by the defining relation (12:2) of ζ, all terms in the right-hand member of (12:3) must vanish; therefore, we must have $\zeta' = 0$. Moreover, in this case, the vertices x_i are in normal order with respect to β as well as with respect to α; therefore, we have

$$[\varphi\psi]_\beta = [\varphi\psi]_\alpha = \varphi(x_0, \cdots, x_m)\,\psi(x_m, \cdots, x_{m+n}).$$

Thus, the difference $[\varphi\psi]_\beta - [\varphi\psi]_\alpha$ vanishes, and Relation (12:1) is valid.

In the remaining three cases, two (necessarily consecutive) variables x_i and x_{i+1} will take on the values v_r and v_{r+1} respectively.

Case 2. $x_{m-1} = v_r$; $x_m = v_{r+1}$.

In this case, the first $m + 1$ terms in the right-hand member of (12:3) are zero, since it is only for $i > m$ that the $(m - 1)^{\text{st}}$ and m^{th} variables in $\zeta(x_0, \cdots, \hat{x}_i, \cdots, x_{m+n})$ have the respective determinations v_r and v_{r+1}. We therefore have

$$(12:4)\quad \begin{aligned} &\zeta'(x_0, \cdots, x_{m+n}) \\ &= (-1)^{m+1}\varphi(x_0, \cdots, x_{m-1}, x_m) \sum_{m+1}^{m+n} (-1)^i \psi(x_{m-1}, x_m, \cdots, \hat{x}_i, \cdots, x_{m+n}). \end{aligned}$$

On the other hand,

$$\begin{aligned} [\varphi\psi]_\beta\,(x_0, \cdots, x_{m+n}) &= -\,[\varphi\psi]_\beta\,(x_0, \cdots, x_m, x_{m-1}, \cdots, x_{m+n}) \\ &= -\,\varphi(x_0, \cdots, x_m, x_{m-1})\,\psi(x_{m-1}, x_{m+1}, \cdots, x_{m+n}) \\ &= \varphi(x_0, \cdots, x_m)\,\psi(x_{m-1}, x_{m+1}, \cdots, x_{m+n}), \end{aligned}$$

and

$$[\varphi\psi]_\alpha\,(x_0, \cdots, x_{m+n}) = \varphi(x_0, \cdots, x_m)\,\psi(x_m, x_{m+1}, \cdots, x_{m+n})\,.$$

Therefore

$$\begin{aligned}(12{:}5)\qquad &[\varphi\psi]_\beta - [\varphi\psi]_\alpha \\ &= \varphi(x_0, \cdots, x_m)\,\{\psi(x_{m-1}, x_{m+1}, \cdots, x_{m+n}) - \psi(x_m, x_{m+1}, \cdots, x_{m+n})\} \\ &= (-1)^m\,\varphi(x_0, \cdots, x_m) \sum_{m-1}^{m} (-1)^i\,\psi(x_{m-1}, \cdots, \hat{x}_i, \cdots, x_{m+n})\,.\end{aligned}$$

Now, if we substract this last expression from (12:4) we have

$$\begin{aligned}&\zeta' - \{[\varphi\psi]_\beta - [\varphi\psi]_\alpha\} \\ &\qquad = (-1)^{m+1}\,\varphi(x_0, \cdots, x_m) \sum_{m-1}^{m+n} (-1)^i\,\psi(x_{m-1}, \cdots, \hat{x}_i, \cdots, x_{m+n}) \\ &\qquad = (-1)^{m+1}\,\varphi(x_0, \cdots, x_m)\,\psi'(x_{m-1}, \cdots, x_{m+n})\,,\end{aligned}$$

which vanishes because the derivative ψ' of the exact function ψ must be zero.

Case 3. $x_m = v_r;\ x_{m+1} = v_{r+1}$.

This case is essentially like the last. In the right-hand member of (12:3) all terms after the m^{th} are zero. We therefore have

$$\begin{aligned}&\zeta'(x_0, \cdots, x_{m+n}) \\ &\qquad = (-1)^{m+1} \sum_{0}^{m-1} (-1)^i\,\varphi(x_0, \cdots, \hat{x}_i, \cdots, x_{m+1})\,\psi(x_m, x_{m+1}, \cdots, x_{m+n})\,.\end{aligned}$$

Moreover,

$$[\varphi\psi]_\beta - [\varphi\psi]_\alpha = (-1)^m \sum_{m}^{m+1} (-1)^i\,\varphi(x_0, \cdots, \hat{x}_i, \cdots, x_{m+1})\ \psi(x_m, \cdots, x_{m+n})\,.$$

Therefore

$$\zeta' - \{[\varphi\psi]_\beta - [\varphi\psi]_\alpha\} = (-1)^{m+1}\,\varphi'(x_0, \cdots, x_{m+1})\,\psi(x_m, \cdots, x_{m+n})$$

since φ is exact, and (12:1) is valid.

Case 4. $x_i = v_r;\ x_{i+1} = v_{r+1}$, but $i \neq m-1, m$.

This case is particularly simple. All terms in the right-hand member of (12:3) are zero; therefore, $\zeta' = 0$. Moreover, we have at once $[\varphi\psi]_\beta = [\varphi\psi]_\alpha$. Therefore (12:1) is again valid.

This completes the argument for the finite case. If the arrangement β differs from the arrangement α by more than a single permutation of two consecutive vertices, we can pass from α to β by a sequence of elementary steps, corresponding to each of which we can construct a function of the form of ζ. Let

$\chi(x_0, \cdots, x_{m+n-1})$ be the sum of these functions. Then we shall, of course, have

$$\chi' = [\varphi\psi]_\beta - [\varphi\psi]_\alpha \qquad \text{(on } \Pi) . \tag{12:6}$$

When the number of vertices is infinite, the argument must be slightly modified, since we cannot, in general, pass from an arrangement α to an arrangement β by simple steps of the sort described above. Let us observe, however, that if we take any finite subset x_i, $(i = 0, 1, \cdots)$, of the vertices v_i arranged in a sequence according to α, we can rearrange them in a sequence according to β by permutations of consecutive vertices. Therefore, if the vertices x_i are the vertices of any finite subcomplex Π_0 of Π, we can construct a function χ such that a relation similar to (12:6) is valid at least on Π_0. The problem will thus be to determine the function χ in such a way that it will be independent of Π_0, in which case Relation (12:6) will be valid over Π as a whole.

We notice that if x_i $(i = 0, 1, \cdots k)$ is any finite sequence of vertices arranged according to α, we can rearrange them according to β by a sequence of steps performed in precisely the following order. Let x_r be the first vertex such that the subsequence $x_0, x_1, \cdots x_r$ is not in normal order with respect to β. Then the two vertices x_{r-1} and x_r must clearly be in inverted order with respect to β. The first step in the rearrangement will be to permute these two vertices. We shall then operate in a similar manner on the arrangement resulting from the first permutation, and so on. Since each permutation reduces by one the number of inversions with respect to β, the process will obviously come to an end after a finite number of steps, with the vertices arranged according to β. Now, the essential point to notice is the following. When we rearrange the sequence x_i according to the above rule, we simultaneously rearrange every subsequence of x_i according to precisely the same rule. For at the moment when we permute two vertices of the complete sequence according to rule, we either leave the arrangement of the subsequence unaltered or change it according to rule, depending on whether or not the two permuted vertices belong to the subsequence.

With the above in view, let us form the complex Π_s consisting of any $(m + n)$-simplex s together with all its components of lower dimensionalities. On the complex Π_s we shall define the function χ in the manner indicated above. That is to say, we shall start with the vertices of Π_s in normal order according to α, rearrange them according to β by applying our rule, and construct the functions ζ corresponding to the various steps in the transition. Moreover, we shall define the function χ on Π_s as the sum of the functions ζ thus constructed. Now, we have only to notice that on any $(m + n - 1)$-component

$$s_i = x_0 \cdots \hat{x}_i \cdots x_{n+m}$$

of s the value of the function ζ corresponding to any given step is zero unless the two vertices that are permuted at this particular step both belong to s_i.

The value of χ on s_i is, therefore, equal to the sum of the functions ζ corresponding to the rearrangement of the vertices of s_i alone. In other words, the value of χ on s_i is independent of the $(m + n)$-simplex s of which s_i is a component. Hence, the function χ is uniquely determined.

V. The Connectivity Ring of a Space

13. An m-function φ will be said to be *derived* with reference to a symbolic space if it is derived with reference to any subcomplex Π_a of the space; it will be said to be *exact* with reference to the space if it is exact with reference to any complex Π_b of the space. With this understanding, the entire discussion, beginning with Theorem 3, will be applicable to symbolic spaces as well as to symbolic complexes. However, the ring of a symbolic space has a natural topology which we shall discuss in a subsequent paper. We shall also discuss the dual relation between the connectivity groups as here defined and the connectivity groups of Vietoris-Čech.

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GENERALIZED RIEMANN MATRICES AND FACTOR SETS

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INTRODUCTION

(*for the expert only*!)

Led by a normalization of integrals of the first kind on a Riemann surface, differing from Riemann's own method in its independence of any dissection of the surface, I suggested in a previous note[1] a natural generalization of the classical concept of Riemann matrices, and expressed the hope that it would tend to simplify considerably the "existential" part in the establishment of necessary and sufficient conditions. The present paper claims to bear out this promise; it contains a complete and simple solution of the problem in terms of "factor sets." This is another point that I had in mind when I drafted the first note: suspicion that it might not be wise under all circumstances to restrict oneself to Galois splitting fields (and thus to sacrifice the minimum degree) induced me to adopt R. Brauer's factor sets in an arbitrary splitting field rather than E. Noether's crossed products over a Galois field. The whole subject is here given a new twist in replacing the study of the commutator algebra $\mathfrak{A}$ of a Riemann matrix R by what I call the associated rational algebra $\mathfrak{L}$; it has the same rational commutator algebra $\mathfrak{A}$ as R and its closure in the field of all real numbers contains R. This change of view was suggested to me by the fact that in the Schur-Brauer theory, construction of splitting fields for a simple algebra $\mathfrak{L}$ is tied up with the maximum subfields of its commutator algebra $\mathfrak{A}$ rather than of $\mathfrak{L}$ itself. Reward in the form of further simplification appears to confirm the new standpoint as a better start for the attack on our problem. To make the paper as easy reading as possible its larger part is devoted to a proof-documented restatement of the foundations,—including the classical facts about simple algebras and the more elementary parts of the Schur-Brauer theory of factor sets.

A. A. Albert's new thorough investigation of the subject[2] unfolding all its aspects and culminating in a complete structural analysis of the commutator algebras of generalized Riemann matrices (Theorems 27–30 on pp. 917–919)

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[1] "On generalized Riemann matrices," *Annals of Math.* **35** (1934), pp. 714–729.—Corrigendum: The formula $\check{P}_{\alpha\beta} = y_{\alpha\beta}P_{\alpha\beta}$ at the end of the last line on p. 724 should read: $C_\alpha^{-1}\check{P}_{\alpha\beta}C_\beta = y_{\alpha\beta}P_{\alpha\dot{\beta}}$.

[2] *Annals of Math.* **36** (1935), pp. 886–964.—Albert does me the honor of associating my name with the kind of matrices under investigation; but would it not be better, if a proper name is to be attached to them, to stand by the former designation as Riemann matrices —even at the expense of having to add an adjective like "generalized"?

prompted me to resume my own line of approach. Though independent in other regards, I borrow from him one essential remark: that apart from a relatively harmless adjunction, the splitting field is totally real; it follows immediately from Rosati's theorem contending that the roots of an "even" matrix of the commutator algebra $\mathfrak{A}$ are real. Its former application to the centrum only was not exhaustive enough. The "harmless" adjunction consists either of nothing, or of a square root, or a quaternion. So it seems natural to distinguish three cases instead of making the old discrimination between "first and second kind" which referred to the character of the centrum.

[Paragraphs included in bold-face square brackets **[ ]** are of minor importance or interrupt the main trend of thought.]

1. Matrix Algebras and their Commutators

(1.1) Let a reference field k be given. A *linear k-set* or *vector-space in k* consists of elements that allow addition and multiplication by numbers in k. n elements $e_1, \cdots, e_n$ such that each element x is expressible as a linear combination

$$x = \xi_1 e_1 + \cdots + \xi_n e_n \qquad (\xi_i \text{ in } k)$$

in a unique manner form a *base* or a coordinate system; the numbers ξ_i are the coordinates of x. The number n is called the *order* of our linear set $\mathfrak{l}$, or the dimensionality of the vector space.

$\mathfrak{l}$ is a *k-algebra* if multiplication of its elements is added to the list of permissible operations.

A square matrix

$$A = \| \alpha_{ij} \| \qquad (i, j = 1, \cdots, g)$$

of g rows and columns is called of *degree g*; the same name applies to a set $\mathfrak{A}$ of matrices A of degree g. A lies in k when all the numbers α_{ij} lie in k. Each such matrix may be interpreted as a linear mapping of a g-dimensional vector space $\mathbf{P}$ on itself. $E = E_g$ denotes the unit matrix of degree g, $A' = \| \alpha_{ji} \|$ the transposed matrix of A. A set $\mathfrak{A} = \{A\}$ goes into an equivalent set if all its mappings A become expressed in a new coordinate system. The set is *irreducible* (in k) if the k-vector space $\mathbf{P}$ contains no linear subspace invariant under all the transformations A of $\mathfrak{A}$, other than 0 and $\mathbf{P}$ itself. $\mathfrak{A}$ is *linearly* or *algebraically closed in k* provided its elements A form a linear k-set or a k-algebra respectively. Any given set $\mathfrak{A}$ gives rise to a linear or algebraic k-closure: the smallest linear k-set, or k-algebra of matrices containing all the members A of $\mathfrak{A}$; it is easy to describe how to construct them if a base of $\mathfrak{A}$ is given. By looking upon the members A of an algebraically closed set $\mathfrak{A}$ as abstract elements allowing addition and multiplication among each other and multiplication by numbers in k, the set $\mathfrak{A}$ changes into an abstract algebra $\mathfrak{a}$ with elements a of which $\mathfrak{A}$; $a \to A$ is a faithful representation. We shall stick to this convention throughout, that corresponding types like A and a, $\mathfrak{A}$ and $\mathfrak{a}$ of the upper case

and the lower case are used to mark the transition from matrices to abstract elements. In general, a correspondence $a \to T(a)$ established between the elements a of an algebra $\mathfrak{a}$ and matrices $T(a)$ in k of degree g is called a *k-representation* of $\mathfrak{a}$ provided it preserves the fundamental operations:

$$T(a + b) = T(a) + T(b) \; ; \qquad T(\lambda a) = \lambda \cdot T(a) \; ; \qquad T(ab) = T(a) \cdot T(b)$$
$$(a, b \text{ elements in } \mathfrak{a} \, , \qquad \lambda \text{ a number in } k) \; .$$

The representation is *faithful* if different elements a are represented by different matrices $T(a)$. The *regular representation* $(\mathfrak{a})$: $a \to (a)$ associates with the element a of $\mathfrak{a}$ the mapping

$$(a): \quad x \to x' = ax$$

whose argument x varies over $\mathfrak{a}$; its representation space is thus $\mathfrak{a}$ itself considered as an h-dimensional vector space $\mathfrak{o}$; its degree is the order h of $\mathfrak{a}$.

A linear k-set or a k-algebra $\mathfrak{A}$ of matrices may be closed in or *extended* to a field K over k; if $e_i (i = 1, \cdots, h)$ is a basis of the set the extended K-set or K-algebra consists of all sums $\sum_i \xi_i e_i$ in which now the components ξ_i vary over K. This operation can thus be described in terms of the abstract scheme $\mathfrak{a}$ (in the case of an algebra, the multiplication table of the e_i is preserved). The extension is denoted by $\mathfrak{A}_K$, $\mathfrak{a}_K$ respectively.

The matrix A is a *commutator* of a given matrix set $\mathfrak{L}$ if it commutes with every member L of $\mathfrak{L}$:

$$AL = LA \, .$$

Those commutators A that lie in a field k form a k-algebra $\mathfrak{A}$ of matrices, the *commutator algebra* in k.

A k-algebra $\mathfrak{A}$ of matrices A in k (or its abstract scheme $\mathfrak{a}$) is a *division algebra* when all its matrices A are non-singular: det $A \neq 0$, with the exception only of $A = 0$. According to Schur's lemma, a matrix A in k that commutes with all matrices of a k-irreducible set $\mathfrak{L}$ is either 0 or non-singular; hence *the commutator algebra $\mathfrak{A}$ in k of the irreducible $\mathfrak{L}$ is a division algebra.* Proof: the columns of the commutator A when considered as vectors span an invariant subspace.

A set $\mathfrak{A}$ of matrices A in k, when irreducible and algebraically closed in k, or its abstract counter-image $\mathfrak{a}$, is called *simple*. A simple algebra $\mathfrak{a}$ shall thus always be defined by means of its faithful irreducible representation $\mathfrak{A}$.

We exclude throughout this section any "degenerate" matrix algebra $\mathfrak{A}$ (or representation) whose matrices A map the total vector space upon the same *proper* linear subspace.

(1.2) About division and simple algebras we remind the reader of the following propositions whose proofs shall here be arranged in as elementary a way as possible and according to two principles: first, the matrix algebras are considered the primary subject, the abstract schemes merely as secondary tools to facilitate their management; second, we shun the somewhat unpleasant "radicals" (there are none in our matrix communities—so why talk about them?).

The terms "matrix," "algebra," "irreducible" refer to a given number field k throughout, and thus mean "matrix in k," "k-algebra" and "irreducible in k."

THEOREM (1.2-A). *A division algebra* $\mathfrak{a}$ *is characterized by these two properties: it contains a unit element* e (satisfying $xe = ex = x$ for all x in $\mathfrak{a}$), *and every element* a *except* 0 *has an inverse* a^{-1}: $a \cdot a^{-1} = a^{-1} \cdot a = e$.

The regular representation $(\mathfrak{a})$ *of* $\mathfrak{a}$ *is faithful as well as irreducible, and hence* $\mathfrak{a}$ *is simple. Each representation* $a \to A(a)$ *is a multiple* t *of* $(\mathfrak{a})$, i.e. in an appropriate coordinate system common to all elements a, the matrix $A(a)$ decomposes into t matrices (a) along the main diagonal.

PROOF. A matrix A of degree g has its characteristic polynomial

$$\varphi(z) = \det(zE - A) = z^g + \alpha_1 z^{g-1} + \cdots + \alpha_g .$$

A itself satisfies the equation

$$\varphi(A) \equiv A^g + \alpha_1 A^{g-1} + \cdots + \alpha_{g-1} A + \alpha_g E = 0 .$$

When A is non-singular, the last coefficient α_g is $\neq 0$. Hence a matrix algebra $\mathfrak{A}$ containing a non-singular element A involves E and the inverse matrix A^{-1} of A:

$$E = -\frac{1}{\alpha_g}(A^g + \alpha_1 A^{g-1} + \cdots + \alpha_{g-1} A),$$

$$A^{-1} = -\frac{1}{\alpha_g}(A^{g-1} + \alpha_1 A^{g-2} + \cdots + \alpha_{g-1} E) .$$

This remark shows that a division algebra possesses the two properties mentioned in the first paragraph of our theorem after we once and for all have excluded the "trivial case" of the zero-algebra consisting of the one matrix 0.

Vice versa: let $\mathfrak{a} = \{a\}$ have these two properties. We represent a by the linear mapping

$$(a): \quad x \to x' = ax .$$

The terms invariant, irreducible, in the space $\mathfrak{o}$ of the regular representation, shall always refer to the set $(\mathfrak{a})$ of all these transformations (a). In our case the equation $x' = ax$ $(a \neq 0)$ establishes a one-to-one correspondence $x \to x'$ (inversion $x = a^{-1}x'$) and hence (a) is non-singular. The regular representation is faithful since (a) and (b) carry the unit element e into two different elements $ae = a$ and $be = b$ if $a \neq b$. Hence, in replacing $\mathfrak{a}$ by the matrix algebra $(\mathfrak{a})$, our original definition of a division algebra is fulfilled.

$(\mathfrak{a})$ is irreducible. An invariant subspace of $\mathfrak{o}$ when containing a single element $i \neq 0$ necessarily involves all elements of form ai (a in $\mathfrak{a}$) and hence every element b of $\mathfrak{a}$ whatsoever $(a = b \cdot i^{-1})$. This proves $\mathfrak{a}$ to be simple.

Suppose, finally, we are given an arbitrary representation $a \to A = A(a)$ of $\mathfrak{a}$ in an n-dimensional vector space $\mathbf{P}$ whose generic vector is denoted by $\mathfrak{x}$ and which we span by a coordinate system $\mathfrak{e}_1, \cdots, \mathfrak{e}_n$. The terms invariant, irreducible, when applied to subspaces of $\mathbf{P}$, refer to the algebra $\mathfrak{A}$ of matrices A.

An equation $\mathfrak{x}' = a\mathfrak{x}$ is to be interpreted as meaning $\mathfrak{x}' = A(a)\mathfrak{x}$. Let $\mathbf{P}_i$ be the subspace consisting of all vectors $\mathfrak{x} = x\mathfrak{e}_i$, one obtains when x varies over $\mathfrak{a}$. The correspondence $x \rightarrow \mathfrak{x}$ thus established is a similarity, i.e. ax goes into $a\mathfrak{x}$; hence $\mathbf{P}_i$ is invariant under the transformations A of $\mathfrak{A}$. Either $\mathbf{P}_i$ is zero or this mapping of $\mathfrak{o}$ on $\mathbf{P}_i$ is a one-to-one correspondence. Indeed, the elements x for which $x\mathfrak{e}_i = 0$ form an invariant subspace of $\mathfrak{o}$; and as $(\mathfrak{a})$ is irreducible, either every x or no x except zero, satisfies $x\mathfrak{e}_i = 0$. (This "typical argument" recurs again and again.) In taking up the subspaces $\mathbf{P}_1, \cdots, \mathbf{P}_n$ one after the other, a $\mathbf{P}_i$ is either contained in the sum of the preceding ones, or linearly independent of them; this fact is just another application of the typical argument. By dropping a term $\mathbf{P}_i$ in the first case one reduces our sequence $\mathbf{P}_1, \cdots, \mathbf{P}_n$ to a decomposition of $\mathbf{P}$ into linearly independent irreducible invariant subspaces in each of which $\mathfrak{A}$ induces a representation equivalent to $(\mathfrak{a})$.

THEOREM (1.2-B). *A simple algebra* $\mathfrak{a}$ *contains a unit element. Its regular representation is a multiple t of that faithful irreducible representation* $\mathfrak{A}$: $a \rightarrow A$ *through which* $\mathfrak{a}$ *was defined. The order h is a multiple of the degree g*: $h = gt$.

The matrices A of degree g are linear mappings in a g-dimensional vector space $\mathbf{P}$. The regular representation $(\mathfrak{A})$ associates with A the linear mapping

$$(A): \quad X \rightarrow X' = AX$$

whose argument X varies within the linear set $\mathfrak{A}$ that here appears as an h-dimensional vector space $\mathfrak{o}$. Let us pick out an irreducible invariant subspace $\mathfrak{o}_1$ of $\mathfrak{o}$. $\mathfrak{o}_1$ is similar to $\mathbf{P}$ under their respective transformations (A) and A. Indeed, let A^0 be an element $\neq 0$ in $\mathfrak{o}_1$ and $\mathfrak{e}$ a vector in $\mathbf{P}$ such that $A^0\mathfrak{e} \neq 0$. The formula $\mathfrak{x} = X\mathfrak{e}$ (X in $\mathfrak{o}_1$) maps $\mathfrak{o}_1$ on an invariant subspace $\mathfrak{o}_1\mathfrak{e}$ of $\mathbf{P}$ by the similarity $X \rightarrow \mathfrak{x}$; for $X \rightarrow \mathfrak{x}$ entails $AX \rightarrow A\mathfrak{x}$. The subspace $\mathfrak{o}_1\mathfrak{e}$ is either zero or the whole space $\mathbf{P}$, because of the irreducibility of $\mathfrak{A}$. The first possibility is here excluded by $A^0\mathfrak{e} \neq 0$. In the remaining case the similarity $X \rightarrow \mathfrak{x}$ is a one-to-one correspondence between $\mathfrak{o}_1$ and $\mathbf{P}$ due to the irreducibility of $\mathfrak{o}_1$. This proves that any irreducible part of $(\mathfrak{A})$ is equivalent to the representation $\mathfrak{A}$. There exists an element I_1 in $\mathfrak{o}_1$ such that $\mathfrak{e} = I_1\mathfrak{e}$. Because of the invariance of $\mathfrak{o}_1$, the matrix XI_1 lies in $\mathfrak{o}_1$ for every matrix X in $\mathfrak{o}$; since both matrices X and XI_1 change $\mathfrak{e}$ into the same vector $\mathfrak{x} = X\mathfrak{e}$ they must coincide for an X lying in $\mathfrak{o}_1$; in particular $I_1I_1 = I_1$. The formula

$$X = XI_1 + (X - XI_1) = X_1 + Y_1$$

decomposes $\mathfrak{o}$ into two independent invariant subspaces: $\mathfrak{o}_1$ with the idempotent generator I_1, and a remainder $\mathfrak{o}_1^*$ consisting of all matrices of the form $Y_1 = X - XI_1$. The elements X_1 and Y_1 of $\mathfrak{o}_1$ and $\mathfrak{o}_1^*$ obey the relations $X_1I_1 = X_1$, $Y_1I_1 = 0$, respectively. Continuation of this process leads to the decomposition of the regular representation into irreducible parts each of which is equivalent to the representation $\mathfrak{A}$:

$$X_1 = XI_1, \qquad Y_1 = X - XI_1; \qquad X_2 = Y_1I_2^*, \qquad Y_2 = Y_1 - Y_1I_2^*;$$

and so on. Hence the regular representation $(\mathfrak{a})$ is a multiple t of $\mathfrak{A}$ and $h = tg$.

We have constructed the decomposition $\mathfrak{o} = \mathfrak{o}_1 + \mathfrak{o}_2 + \cdots + \mathfrak{o}_t$:

$$X = X_1 + X_2 + \cdots = XI_1 + (X - XI_1)I_2^* + \cdots$$
$$= XI_1 + XI_2 + \cdots ;$$
$$I_1 = I_1, \qquad I_2 = I_2^* - I_1I_2^*, \cdots .$$

As XI_α is the component X_α of X lying in $\mathfrak{o}_\alpha$ we have

$$I_\beta I_\alpha = 0 \text{ for } \beta \neq \alpha, \qquad I_\alpha I_\alpha = I_\alpha .$$

The sum

$$I = I_1 + I_2 + \cdots + I_t$$

satisfies the equation $XI = X(X \text{ in } \mathfrak{A})$, in particular $II = I$. All vectors $\mathfrak{y}$ carried by I into zero: $I\mathfrak{y} = 0$, form an invariant subspace of $\mathbf{P}$ because of

$$X\mathfrak{y} = XI\mathfrak{y} = 0 \qquad (X \text{ in } \mathfrak{A}).$$

Hence either all vectors $\mathfrak{y}$ fulfill this equation, or the vector $\mathfrak{y} = 0$ only. The first possibility would result in the trivial case once and for all excluded. Since $\mathfrak{y} = \mathfrak{x} - I\mathfrak{x}$ satisfies the equation $I\mathfrak{y} = 0$, the other alternative leads to the identity $\mathfrak{x} = I\mathfrak{x}$, proving I to be the unit matrix E.

One may add to our theorem the statement that *every k-representation of* $\mathfrak{a}$ *decomposes into irreducible parts equivalent to the representation* $\mathfrak{A}$. This is an immediate consequence of the general proposition:

THEOREM (1.2-C). *If the regular representation* ($\mathfrak{a}$) *of an algebra* $\mathfrak{a}$ *decomposes into irreducible parts* $\mathfrak{A}_1, \mathfrak{A}_2, \cdots$, *then every representation decomposes into parts each of which is equivalent to one of the* $\mathfrak{A}_i$.

PROOF: We assumed that $\mathfrak{a}$, considered as the space $\mathfrak{o}$ of the regular representation, decomposes into irreducible invariant subspaces $\mathfrak{o}_1, \mathfrak{o}_2, \cdots, \mathfrak{o}_t$. Let $\mathfrak{x}$ be the generic vector and $\mathfrak{e}_1, \cdots, \mathfrak{e}_g$ a coordinate system of the space $\mathbf{P}$ of the given representation

$$\mathfrak{A}: \quad a \to A = A(a).$$

Again, $\mathfrak{x}' = a\mathfrak{x}$ shall mean $\mathfrak{x}' = A(a)\mathfrak{x}$ and $\mathfrak{o}_\alpha\mathfrak{e}$ denotes the set of all vectors $\mathfrak{x} = x\mathfrak{e}$ (x in $\mathfrak{o}_\alpha$). We then form the table

$$\mathfrak{o}_1\mathfrak{e}_1, \cdots, \mathfrak{o}_t\mathfrak{e}_1,$$
$$\mathfrak{o}_1\mathfrak{e}_g, \cdots, \mathfrak{o}_t\mathfrak{e}_g.$$

Going through it as one reads the words in a book, and applying essentially the same argument as in the case of the division algebra, we obtain the sought-for decomposition of $\mathbf{P}$.

(1.3) We now pass to the relationship of this analysis to the commutator idea. It springs from the following source:

THEOREM (1.3-A). *If the algebra* $\mathfrak{a}$ *contains a unit element e, the only linear*

transformations that commute with all transformations (a): $x \to x' = ax$ *are of the form* $x \to y = xb$ (b *an element in* $\mathfrak{a}$).

Indeed, if $y = B(x)$ is such a commutator, we must have by definition

$$B(ax) = a \cdot B(x). \tag{1.31}$$

Put $B(e) = b$ and apply (1.31) to $x = e$: one thus gets the formula desired, $B(a) = ab$, for every a.

When we designate by $\mathfrak{a}'$ the *inverse algebra* of $\mathfrak{a}$ differing from $\mathfrak{a}$ in that the product of two elements a and b is now defined as ba rather than ab, we may express our result thus: *The commutator algebra of the regular representation of* $\mathfrak{a}$ *is the regular representation of* $\mathfrak{a}'$; the relationship is hence *mutual.*

This applies in particular to a division algebra $\mathfrak{a}$; then both regular representations $(\mathfrak{a})$ and $(\mathfrak{a}')$ are irreducible.

We take up again our *simple algebra* $\mathfrak{A}$ or $\mathfrak{a}$. The commutator algebra $\mathfrak{B}$ of $\mathfrak{A}$ is *in abstracto* a division algebra $\mathfrak{d}$ (of order d), hence *in concreto* a multiple $t(\mathfrak{d})$ of $\mathfrak{d}$'s regular representation $(\mathfrak{d})$: the generic matrix of $\mathfrak{B}$ has the form

$$\left\| \begin{matrix} B & & \\ & \ddots & \\ & & B \end{matrix} \right\| \quad (t \text{ rows})$$

where B varies over all the operators

$$(b): \quad x \to x' = bx \quad (x \text{ variable in } \mathfrak{d})$$

belonging to the elements b of $\mathfrak{d}$. Hence $g = d \cdot t$. The commutator algebra $\mathfrak{A}^*$ of $\mathfrak{B}$ consists of all matrices of the form

$$\left\| \begin{matrix} A_{11} & \cdots & A_{1t} \\ \cdots & \cdots & \cdots \\ A_{t1} & \cdots & A_{tt} \end{matrix} \right\|,$$

where each A_{ik} is an operator

$$x \to x' = xb \quad (b \text{ in } \mathfrak{d})$$

of the regular representation $(\mathfrak{d}')$ of the inverse division algebra $\mathfrak{d}'$. This we express by the equation

$$\mathfrak{A}^* = (\mathfrak{d}')_t. \tag{1.32}$$

$\mathfrak{A}^*$ evidently contains $\mathfrak{A}$. The fact that *it does not extend beyond* $\mathfrak{A}$ can be established by the following simple indirect argument. Were $\mathfrak{A}^*$ really larger than $\mathfrak{A}$, then the same would be true for any multiple of $\mathfrak{A}$, in particular for the regular representation $(\mathfrak{a})$ of $\mathfrak{a}$ contrary to Theorem (1.3-A), which shows that $(\mathfrak{a})$ and $(\mathfrak{a}')$ are *mutual* commutators. Thus we are enabled to replace the equality (1.32) by *Wedderburn's theorem*:*

$$\mathfrak{A} = (\mathfrak{d}')_t. \tag{1.33}$$

*This shortcut to Wedderburn's theorem was pointed out to me by R. Brauer.

THEOREM (1.3-B). *The relationship of a simple matrix algebra $\mathfrak{A}$ and its commutator algebra $\mathfrak{B}$ is mutual: $\mathfrak{A}$ is the full commutator algebra of $\mathfrak{B}$. $\mathfrak{B}$ is expressed in terms of a division algebra $\mathfrak{d}$ of order d as $t\cdot(\mathfrak{d})$, $\mathfrak{A}$ as $(\mathfrak{d}')_t$. Besides $h = tg$ we have $g = dt$, hence $h = dt^2$.*

From this follow two important consequences:

THEOREM (1.3-C). (Burnside.) *An irreducible $\mathfrak{A}$ of degree g whose only commutators are multiples αE of the unit matrix E (case $d = 1$) contains g^2 independent matrices (and is therefore irreducible in any field K over k; "absolute irreducibility").*

THEOREM (1.3-D). (*Criterion for irreducibility preserved.*) *The $\mathfrak{A}$ irreducible in k stays irreducible in a field K over k if its commutator algebra in K (as well as in k) is a division algebra.*

Indeed, our equation

$$\mathfrak{A} = (\mathfrak{d}')_t$$

at once leads to

$$\mathfrak{A}_K = (\mathfrak{d}'_K)_t$$

for the extensions to K. Under the assumption that $\mathfrak{d}_K$ and hence $\mathfrak{d}'_K$ is a division algebra, its regular representation $(\mathfrak{d}'_K)$ is irreducible in K and then so is $(\mathfrak{d}'_K)_t$.—The necessity of our criterion which has thus been shown to be sufficient is warranted by Schur's lemma.

The full reciprocity between algebra and commutator algebra is not reached before we pass from the irreducible representation $\mathfrak{A}$ of our simple algebra $\mathfrak{a}$ to a multiple $s\mathfrak{A}$. For this algebra $s\cdot(\mathfrak{d}')_t$ we readily find $t\cdot(\mathfrak{d})_s$ as its commutator algebra. The structure of the generic elements of our two algebras is indicated by the schemes

$$(1.34)\qquad \left|\begin{array}{c|c|c} \begin{matrix} A_{11} & \cdots & A_{1t} \\ \cdots & \cdots & \cdots \\ A_{t1} & \cdots & A_{tt} \end{matrix} & 0 & \\ \hline 0 & \begin{matrix} A_{11} & \cdots & A_{1t} \\ \cdots & \cdots & \cdots \\ A_{t1} & \cdots & A_{tt} \end{matrix} & \\ \hline & & \end{array}\right| \qquad \left|\begin{array}{c|c|c} \begin{matrix} B_{11} & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & B_{11} \end{matrix} & \begin{matrix} B_{12} & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & B_{12} \end{matrix} & \\ \hline \begin{matrix} B_{21} & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & B_{21} \end{matrix} & \begin{matrix} B_{22} & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & B_{22} \end{matrix} & \\ \hline & & \end{array}\right|$$

where all A_{ik} vary independently in $(\mathfrak{d}')$, all $B_{\alpha\beta}$ in $(\mathfrak{d})$; $i, k = 1, \cdots, t$; $\alpha, \beta = 1, \cdots, s$.

THEOREM (1.3-E). *A representation $\mathfrak{A}$ of a simple algebra has as its commutator a matrix algebra $\mathfrak{B}$ of the same type. The relationship is mutual: $\mathfrak{A}$ is the commutator algebra of $\mathfrak{B}$. More exactly, the structure is described by*

$$\mathfrak{A} = s\cdot(\mathfrak{d}')_t, \qquad \mathfrak{B} = t\cdot(\mathfrak{d})_s$$

where $\mathfrak{d}$ *is an (abstract) division algebra of order* d. *The degree of* $\mathfrak{A}$ *and* $\mathfrak{B}$ *equals* $d \cdot st$, *order of* $\mathfrak{A} = d \cdot t^2$, *order of* $\mathfrak{B} = d \cdot s^2$.

[The appendix 7 treats the automorphisms of algebras $\mathfrak{A}$ of the type here considered. Though we have no need for the facts there expounded, they follow so easily and naturally from our considerations that I could not resist the temptation of completing my account by their statement and proof.]

(1.4) Our next concern is a natural generalization of matrix algebras: the elements a may be n-uples

$$a = (A_1, A_2, \cdots, A_n)$$

of matrices in k, each component A_i being a matrix of prescribed degree g_i. Such elements may be added and multiplied among each other and multiplied by numbers in k by performing these operations on the several components separately. We want to study algebras $\mathfrak{a}$ in k consisting of such elements a. Each component like $A_1 = A_1(a)$ defines a representation $\mathfrak{A}_1$ of $\mathfrak{a}$: $a \rightarrow A_1$. The second part of Schur's lemma states that every matrix B in k satisfying the relation

$$A_1(a)B = BA_2(a)$$

identically in a must be zero provided the two component representations $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are irreducible and inequivalent. We prove:

THEOREM (1.4-A). *If the component representations of an n-uple matrix algebra* $\mathfrak{a}$ *are irreducible and inequivalent, then the n components* A_i *are independent of each other. (The regular representation of* $\mathfrak{a}$, *and hence every representation, is decomposable into irreducible parts each equivalent to one of the component representations.)*

The asserted "independence" may be formulated in different manners; the simplest formulation is perhaps as follows: if

$$a = (A_1, A_2, \cdots, A_n)$$

is contained in $\mathfrak{a}$, then the same holds for

$$\text{(1.41)} \qquad \begin{aligned} a_1 &= (A_1, 0, \cdots, 0)\,, \\ &\cdots\cdots\cdots\cdots\cdots \\ a_n &= (0, 0, \cdots, A_n)\,. \end{aligned}$$

Or: with a varying over $\mathfrak{a}$, each component $A_i(a)$ varies *independently* over its whole range $\mathfrak{A}_i$; or: $\mathfrak{a}$ is the direct sum of the algebras $\mathfrak{A}_i$.

The proof follows exactly the lines laid out in the proof of Theorem (1.2-B). In an irreducible invariant subspace $\mathfrak{r}_1$ of $\mathfrak{r}$ we picked out an element $a^0 \neq 0$. At least one of its n components A_i^0, let us say A_1^0, is $\neq 0$. We then chose a vector $\mathfrak{e}$ such that $A_1^0\, \mathfrak{e} \neq 0$, and concluded that $\mathfrak{r}_1$ is similar to the first component space, i.e. the representation space of $\mathfrak{A}_1$ (or that the representation induced by the regular one in $\mathfrak{r}_1$ is equivalent to $\mathfrak{A}_1$). We now add this little remark: For no element a in $\mathfrak{r}_1$ can the second component A_2 be $\neq 0$. For

then, starting with such an a instead of a^0 we should find that $\mathfrak{g}_1$ is similar to the second component space, which is impossible because of the inequivalence of $\mathfrak{A}_1$ and $\mathfrak{A}_2$. After the decomposition of $\mathfrak{g}$ into irreducible invariant subspaces $\mathfrak{g}_1, \mathfrak{g}_2, \cdots$ we unite those that are similar to the first component space, those similar to the second component space, and so on, and we thus arrive at the desired decomposition into independent components of form (1.41).

We finally consider a k-algebra $\mathfrak{a}$ of matrices in k which is decomposable into irreducible parts. Writing the equivalent ones among them alike, the generic element a breaks up into "blocks" of the kind:

$$\boxed{\begin{matrix} A_i(a) & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & A_i(a) \end{matrix}} \qquad (i = 1, \cdots v),$$

where

$$\mathfrak{A}_i : a \to A_i(a)$$

are irreducible and mutually inequivalent representations. The second part of Schur's lemma shows that each commutator breaks up into blocks of the same size. Together with our proposition concerning the independence of the several blocks in a, this leads to the culminating result[3] of our whole investigation:

THEOREM (1.4-B). *If a k-algebra $\mathfrak{A}$ of matrices in k is decomposable into irreducible parts, so is its commutator algebra $\mathfrak{B}$. $\mathfrak{A}$ is conversely the commutator algebra of $\mathfrak{B}$. Their structure is described by formulas*

$$\mathfrak{A} = \sum_{i=1}^{v} s_i(\mathfrak{d}_i')_{t_i}, \qquad \mathfrak{B} = \sum_{i=1}^{v} t_i(\mathfrak{d}_i)_{s_i}$$

where $\mathfrak{d}_i$, $\mathfrak{d}_i'$ are inverse (abstract) division algebras.

2. The Associated Linear Set and Algebra of a Riemann Matrix

(2.1) Two fields play a decisive part for Riemann matrices: the field k of *rational numbers* and that K of *real numbers*. One may replace k by any "real" field in the sense of the Artin-Schreier theory,[4] and K by a really closed real field over k; a real field k is of characteristic 0. No peculiar traits beyond that shall be made use of in our discussions, but it is pleasant to be able to refer to numbers in k and K respectively as "rational" and "real" numbers.

Let C be a symmetric or skew-symmetric non-singular rational matrix of

[3] Attributed to Rabinowitsch by v. d. Waerden, *Gruppen von linearen Transformationen*, Berlin, 1935, p. 53.

[4] *Abhandlungen Math. Sem. Hamburg*, vol. **5** (1926), pp. 85–99.

degree g, and $S = \| s_{ij} \|$ a symmetric real and positive-definite matrix of the same degree, i.e. one whose corresponding quadratic form

$$\sum_{i, j=1}^{g} s_{ij} x_i x_j$$

of the g real variables x_i is positive-definite. Then

$$R = C^{-1} S \tag{2.11}$$

is called a *(generalized) Riemann matrix*. The two cases $C' = \pm C$ are distinguished by the attribute *even* or *odd*. If the rational matrix A commutes with R, the Riemann matrix R is said to allow the *complex multiplication* A (it would probably be better to substitute the word "matric" for complex). About the significance of this concept for Riemann surfaces and their integrals, the necessary information is to be found in my note referred to above; we are concerned with the natural generalization of the problem of complex multiplication for elliptic functions from the genus 1 to arbitrary genus.

By a transformation U with rational coefficients one may introduce a new "rational" coördinate system in the underlying vector space. R is then changed into the equivalent $U^{-1}R\,U$, whereas C and S are to be transformed according to:

$$C \to U' C U, \qquad S \to U' S U .$$

The relation (2.11) or

$$CR = S \tag{2.12}$$

as well as the symmetries

$$C' = \pm C, \qquad S' = S \tag{2.13}$$

are then preserved. Later on we shall have occasion to use other "real" coördinate systems besides the rational ones. The positive-definite character of S has the consequence that (in an arbitrary real coördinate system) if we cut S:

$$S = \begin{Vmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{Vmatrix}$$

then not only S but the principal minors S_{11}, S_{22} as well are non-singular (and positive-definite).

(2.2) The first step one can take is to substitute for R the *smallest linear k-set Λ of matrices in k whose extension Λ_K to K contains R.* I call Λ the *associated linear set.* It provides the most complete reagent for the rational properties of R; for it exhibits them all while automatically extinguishing the transcendental features of R which the algebraist is so anxious to forget about. Two Riemann matrices whose associated linear sets are (rationally) equivalent may therefore be named *kindred* matrices. This closest rational kinship by no means implies the rational equivalence of the Riemann matrices

themselves. The existence of a common cross-cut Λ of all linear k-sets of matrices in k, whose extension to K contains R, is established by the following considerations.

Let $L_1, \cdots, L_l$ and $M_1, \cdots, M_m$ be the bases of two such linear k-sets Λ and $\mathbf{M}$:

$$R = x_1^0 L_1 + \cdots + x_l^0 L_l = y_1^0 M_1 + \cdots + y_m^0 M_m$$

(x_i^0, y_k^0 real numbers). The solutions $(x_i; y_k)$ of the linear equations

$$x_1 L_1 + \cdots + x_l L_l = y_1 M_1 + \cdots + y_m M_m$$

with rational coefficients have a base consisting of *rational* solutions. When we express the particular solution $(x_i^0; y_k^0)$ as a linear combination of them, we express R as a linear combination of matrices common to Λ and $\mathbf{M}$.

Some obvious properties of the associated set Λ of base $L_1, \cdots, L_l$ are readily ascertained. The symmetry of (2.12) together with $C' = \pm C$ yields

$$R' C = \pm C R. \tag{2.21}$$

From every matrix L of Λ we form L_* by

$$L_*' = C L C^{-1}. \tag{2.22}$$

The extension to K of the linear set Λ_* thus obtained, involves R according to (2.21); hence $\Lambda \prec \Lambda_*$ and then $\Lambda = \Lambda_*$ because the order of Λ_* equals that of Λ; or the linear process $L \rightarrow L_*$ carries each L of Λ into an L_* of Λ again. (2.22) may be written in both forms:

$$L_*'C = CL \quad \text{or} \quad CL_* = L'C \tag{2.23}$$

owing to $C' = \pm C$. Hence the same operation $L \rightarrow L_*$ carries L_* back into L and is therefore an *involution.* A rational commutator A of R is at the same time a commutator of Λ. Indeed, the solutions x_i of the rational linear equations $AL = LA$ for the generic element L of Λ_K:

$$L = x_1L_1 + \cdots + x_lL_l$$

have a rational base. We thus determine a linear subset within Λ whose elements L satisfy $AL = LA$ and whose extension to K includes R. The minimum property of Λ requires the subset to exhaust Λ. Adding a remark of similarly obvious nature, we sum up:

Theorem (2.2-A). *All linear k-sets of matrices in k whose extension to K involves R, have a common cross-cut of the same property, $\Lambda = \{L\}$, the associated linear set. Λ allows a linear involution $L \rightarrow L_*$ as defined by* (2.22). *The rational commutators of R and Λ coincide. Any rational reduction of R goes hand-in-hand with a parallel reduction of Λ and vice versa.*

The first non-trivial and encouraging fact about Riemann matrices is Poincaré's theorem of reduction:

THEOREM (2.2-B). *The associated set Λ of a Riemann matrix R is decomposable into irreducible parts.*

PROOF: With respect to a given reduction of $\Lambda = \{L\}$:

$$L = \begin{Vmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{Vmatrix}, \qquad R = \begin{Vmatrix} R_{11} & 0 \\ R_{21} & R_{22} \end{Vmatrix}$$

we write

$$C = \begin{Vmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{Vmatrix}, \qquad S = \begin{Vmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{Vmatrix}$$

$C_{22} R_{22} = S_{22}$ proves C_{22} to be non-singular. We infer from the equation $L_*'C = CL$ for

$$L_*' = \begin{Vmatrix} L_{11}^* & L_{12}^* \\ 0 & L_{22}^* \end{Vmatrix}, \qquad L = \begin{Vmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{Vmatrix}$$

the two relations

$$L_{22}^* C_{22} = C_{22} L_{22} ,$$

$$L_{22}^* C_{21} = C_{21} L_{11} + C_{22} L_{21} .$$

One substitutes $L_{22}^* = C_{22} L_{22} C_{22}^{-1}$ from the first into the second equation, and gets

$$L_{22} C_{22}^{-1} C_{21} - C_{22}^{-1} C_{21} L_{11} = L_{21} .$$

This shows that the rational transformation

$$\begin{Vmatrix} E & 0 \\ B & E \end{Vmatrix}, \qquad B = C_{22}^{-1} C_{21}$$

effects the desired decomposition:

$$\begin{Vmatrix} E & 0 \\ B & E \end{Vmatrix} \cdot \begin{Vmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{Vmatrix} = \begin{Vmatrix} L_{11} & 0 \\ 0 & L_{22} \end{Vmatrix} \cdot \begin{Vmatrix} E & 0 \\ B & E \end{Vmatrix}.$$

After this has been accomplished:

$$L = \begin{Vmatrix} L_1 & 0 \\ 0 & L_2 \end{Vmatrix}, \qquad R = \begin{Vmatrix} R_1 & 0 \\ 0 & R_2 \end{Vmatrix},$$

and the relations

$$C_{11} R_1 = S_{11}, \qquad C_{22} R_2 = S_{22}$$

show that the parts R_1, R_2 are Riemann matrices.

(2.3) There is no machinery ready for handling linear matrix sets. However, when we remember that a matrix A commuting with two matrices L_1 and L_2

also commutes with L_1L_2, we are led to replace Λ by its algebraic closure $\mathfrak{L}$ in k. It arises when we form products of any number of elements of Λ and their linear combinations. $\mathfrak{L}$ is called *the associated algebra of* R, and two Riemann matrices are *associated* when they possess the same or equivalent associated algebras. This "association" is much weaker than the "kinship" before mentioned; many finer rational traits of R are effaced by substituting for Λ its embedding algebra $\mathfrak{L}$—the smallest k-algebra of k-matrices whose extension to K includes R. We thus take refuge in the mathematician's usual makeshift: if one can't solve a problem, one dilutes it so that one can. We have one strong excuse, however, in our case: we retain enough for the treatment of the problem of "matric multiplication." It is evident that the involutorial operation $L \to L_*$ defined by (2.22) takes place within $\mathfrak{L}$ as well as in Λ. Considered as an operation in the abstract algebra $\mathfrak{l}$ it is an involutorial anti-automorphism satisfying the rules

$$(p + q)_* = p_* + q_*, \qquad (\alpha p)_* = \alpha p_*, \qquad (pq)_* = q_*p_*$$

$$(p,\ q \text{ elements in } \mathfrak{l},\ \alpha \text{ a number in } k).$$

An algebra allowing an anti-automorphic involution $p \to p_*$ may be called *involutorial*. The *even* and *odd* elements are those satisfying the equations $p_* = p$, $p_* = -p$ respectively. Each element is the sum of an even and an odd element:

$$p = \tfrac{1}{2}(p + p_*) + \tfrac{1}{2}(p - p_*).$$

R is an even or odd element in the closure $\mathfrak{L}_K$ according as R is an even or odd Riemann matrix.

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The last remark, an immediate consequence of Theorem (1.4-B), affords a new definition of the associated algebra from which its properties could equally easily have been derived, and it shows that $\mathfrak{L}$ and $\mathfrak{A}$ both encompass exactly the same amount of information about the rational nature of R. Since

$$AL = LA \qquad \text{implies} \qquad L'A' = A'L',$$

$\mathfrak{A}$ as well as $\mathfrak{L}$ is involutorial, the involution in $\mathfrak{A}$: $A \to A^*$ being defined by the same equation (2.22):

$$A'_* = CAC^{-1}.$$

Our analysis of the structure of a fully decomposable matrix algebra $\mathfrak{L}$, by warranting the inequivalent irreducible parts to be independent variables, reduces the problem without any loss to the case of a rationally irreducible R and $\mathfrak{L}$. Thus for the rest of the paper we assume R as a *pure*, i.e. irreducible, Riemann matrix. *Our chief problem is to ascertain the necessary and sufficient conditions that a given algebra $\mathfrak{L}$ is a Riemann algebra, namely an algebra associated with some pure Riemann matrix R.*

3. Splitting Fields and Factor Sets, both Absolute and Relative

After the easy advance through open territory, the battle now starts in earnest. We had better put in place, therefore, our big guns: splitting field and factor set.

(3.1) First, some preliminary remarks about algebraic extensions of the reference field k (of characteristic 0).

An irreducible equation $f(x) = 0$ of degree n with coefficients in k determines a field $k(\vartheta)$ of degree n; $f(\vartheta) = 0$. Each number η in $k(\vartheta)$ is of the form

(3.11) $$\eta = e_0 + e_1\vartheta + \cdots + e_{n-1}\vartheta^{n-1} \qquad (e_i \text{ in } k).$$

In some field over k, $f(x)$ breaks up into n different linear factors:

$$f(x) = \prod_{\alpha=1}^{n} (x - \vartheta_\alpha).$$

We have the n *conjugations* $\vartheta \to \vartheta_\alpha$ sending η, (3.11) over into

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The n fields $k(\vartheta_\alpha)$ are, as it were, copies of the model $k(\vartheta)$ which no one can algebraically tell apart; each field does exactly the same as the other, like show girls in a parade. When looked upon as a linear transformation between variables e_i and η_α the Vandermonde transformation (3.12), V_n, is non-singular.

Two pairs of indices (α, β) and (α', β') are *conjugate* when a permutation of the Galois group carries ϑ_α, ϑ_β into $\vartheta_{\alpha'}$, $\vartheta_{\beta'}$; or when each polynomial $F(x, y)$ in k vanishing for $x = \vartheta_\alpha$, $y = \vartheta_\beta$ also vanishes for $\vartheta_{\alpha'}$, $\vartheta_{\beta'}$. A number

$$\eta_{\alpha\beta} = G(\vartheta_\alpha, \vartheta_\beta)$$

in $k(\vartheta_\alpha, \vartheta_\beta)$ then has a definite conjugate $\eta_{\alpha'\beta'} = G(\vartheta_{\alpha'}, \vartheta_{\beta'})$ not affected by what is arbitrary in the choice of the polynomial $G(x, y)$ in k. A double set

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(3.13) $$\eta_{\alpha\beta} = \sum_{i,j=0}^{n-1} e_{ij}\vartheta_\alpha^i\vartheta_\beta^j \qquad (e_{ij} \text{ in } k).$$

Nevertheless our original and less formal definition is preferable in view of its easier management. Analogous definitions apply to triple sets, and so on.

A subfield κ of $k(\vartheta)$ over k of degree v determines a partition of the indices α into v classes Γ of m "*coördinated*" indices each: α and β are called coördinated if $\eta_\alpha = \eta_\beta$ for all numbers η in κ. Any given coördination of the n indices into classes can thus be generated provided coördination is not destroyed by conjugation: whenever α, β are coördinated and the pair (α', β') is conjugate to (α, β), we suppose that then α', β' are coördinated also.

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Nevertheless our original and less formal definition is preferable in view of its easier management. Analogous definitions apply to triple sets, and so on.

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The term "conjugate double set" $\eta_{\alpha\beta}$ (or triple set, and so on) keeps a definite meaning if we assume $\eta_{\alpha\beta}$ to be defined merely for coördinated subscripts α, β; we then speak of a *conjugate set over* κ. Instead of (3.13) it is more convenient to use a representation

$$(3.14) \qquad \eta_{\alpha\beta} = \sum_{i,j=0}^{m-1} e_\Gamma^{ij} \vartheta_\alpha^i \vartheta_\beta^j$$

where e^{ij} is a number in κ and e_Γ^{ij} denotes the conjugate $e_\alpha^{ij} = e_\beta^{ij} = \cdots$ for the class $\Gamma = \alpha, \beta, \cdots$ of coördinated indices.

(3.2) Let $\mathfrak{l}$ be a *simple algebra* in k of order h and $\mathfrak{L}$: $\mathfrak{l} \to L$ its irreducible faithful representation of degree g by which $\mathfrak{l}$ was defined; $h = tg$. I. Schur constructed the *splitting field* in the following manner.[5]

We take a rational commutator A of $\mathfrak{L}$. Since every root ϑ of the characteristic polynomial $\varphi(z)$ of A satisfies the equation $| A - \vartheta E | = 0$ the relation $| \psi(A) | = 0$ holds for every factor $\psi(z)$ of $\varphi(z)$; we suppose that $\psi(z)$ lies in k and is irreducible in k. But then as the commutator algebra $\mathfrak{A}$ of the irreducible $\mathfrak{L}$ is a division algebra, not only the determinant but the matrix $\psi(A)$ itself must vanish. In the g-dimensional vector space $\mathbf{P}$ where A represents a linear substitution, we choose an arbitrary vector $\mathfrak{e} \neq 0$ (with rational coefficients) and form successively

$$\mathfrak{e}_0 = \mathfrak{e}, \qquad \mathfrak{e}_1 = A\mathfrak{e}_0, \qquad \mathfrak{e}_2 = A\mathfrak{e}_1, \cdots.$$

If ψ is of degree n, one derives from the equation $\psi(A) = 0$ and the irreducibility of ψ the fact that the vectors $\mathfrak{e}_0, \mathfrak{e}_1, \cdots, \mathfrak{e}_{n-1}$ span an n-dimensional subspace $\mathbf{P}_1$ of $\mathbf{P}$ which is invariant with respect to A and in which $\psi(z)$ is the characteristic polynomial of A. Starting with a vector $\mathfrak{e}_0'$ not contained in $\mathbf{P}_1$, the same procedure furnishes a second independent subspace $\mathbf{P}_2$ in which the same is true; and so forth. Therefore one must have

$$\varphi(z) = (\psi(z))^f, \qquad g = fn.$$

If

$$\psi(z) = \prod_{\alpha=1}^{n} (z - \vartheta_\alpha)$$

the matrix of the transformation A of $\mathbf{P}_1$, when expressed in the coördinate system $\mathfrak{e}_0, \mathfrak{e}_1, \cdots, \mathfrak{e}_{n-1}$, is changed by the Vandermonde transformation (3.12), V_n, into the diagonal matrix

$$\begin{Vmatrix} \vartheta_1 & & \\ & \ddots & \\ & & \vartheta_n \end{Vmatrix}.$$

[5] *Transactions Amer. Math. Soc.* (2) **15** (1909), p. 159.

The result is this: in an "irrational" coördinate system changing into a rational one by the Vandermonde transformation $V_n \times E_f$, the matrix A appears as the diagonal matrix of the roots

$$\vartheta_1, \cdots, \vartheta_1, \vartheta_2, \cdots, \vartheta_2, \cdots, \vartheta_n, \cdots, \vartheta_n \qquad \text{(each root } f \text{ times)}.$$

Since the ϑ_α are all distinct, the generic element L of $\mathfrak{L}$, since it satisfies the equation $AL = LA$, splits up in the same coördinate system in the following way:

$$\left\| \begin{matrix} L_1 & & \\ & \ddots & \\ & & L_n \end{matrix} \right\| \sim L,$$

where $L_\alpha = L(\vartheta_\alpha)$ are conjugate matrices of degree f in the conjugate fields $k(\vartheta_\alpha)$. ($\sim$ stands for: "changes into . . . by the Vandermonde transformation $V_n \times E_f$.") The L_α form the algebra $\mathfrak{L}_\alpha$; the algebra of the $L(\vartheta)$ may be designated by $\mathfrak{L}(\vartheta)$ if ϑ is any one of the roots ϑ_α, no matter which. $\mathfrak{L}(\vartheta)$ is irreducible in $k(\vartheta)$ because a reduction in $k(\vartheta)$ would result in a rational reduction of $\mathfrak{L}$.

The number field $k(\vartheta)$ is isomorphic to the field $k(a)$ consisting of the polynomials of $a = A$ with coefficients in k. If we have an element $b = B$ in the commutator algebra $\mathfrak{A}$ commuting with a, the splitting can be pushed forward, for then, in the irrational coördinate system just introduced, B decomposes like L into conjugate matrices B_α; B_α is a commutator of $\mathfrak{L}_\alpha$. In applying the above consideration on B_α rather than on A and in the field $k(\vartheta_\alpha)$ instead of k, we bring B_α into diagonal form and obtain a corresponding splitting of L_α. Only then the splitting made no headway when b belongs to the field $k(a)$. In this manner one finally arrives by successive adjunctions of elements a of $\mathfrak{A}$ commutable among each other, at a *maximal* a which has the property that each element b of $\mathfrak{A}$ commutable with a lies in $k(a)$. Our notations shall now refer to such a maximal a; the number field $k(\vartheta)$ isomorphic to $k(a)$ is then called a *splitting field*. Under these circumstances the multiples of the unit matrix are the only matrices commuting with all members $L(\vartheta)$ of the $k(\vartheta)$-irreducible algebra $\mathfrak{L}(\vartheta)$; hence, according to Burnside's theorem, $\mathfrak{L}(\vartheta)$ contains f^2 linearly independent matrices and is *absolutely irreducible*.

[In a more general way the field $k(\eta)$ of degree r is called a splitting field for $\mathfrak{l}$ if $\mathfrak{l}$ allows of an absolutely irreducible representation in $k(\eta)$. One readily concludes from the fact that each representation of $\mathfrak{l}$ is a multiple of the irreducible one $\mathfrak{L}$, that the degree r is a multiple of n. We shall here avail ourselves only of the splitting fields of minimum degree n derived from the commutator algebra.]

(3.3) Two of the conjugate representations $\mathfrak{L}_\alpha$, $\mathfrak{L}_\beta$ may be either equivalent or not. In the second case the equation

$$BL_\beta = L_\alpha B$$

when required to hold for all elements L has by the Schur lemma the only solution $B = 0$, in a field K involving all $k(\vartheta_\alpha)$. In the first case there exists

a non-singular solution B; each solution is a multiple of B and hence either 0 or non-singular. In particular, the equation obviously has a solution $A_{\alpha\beta} \neq 0$ lying in $k(\vartheta_\alpha, \vartheta_\beta)$, if it has a non-vanishing solution at all; $| A_{\alpha\beta} | \neq 0$. We say that the two conjugations $\vartheta \to \vartheta_\alpha$ and $\vartheta \to \vartheta_\beta$ are coördinated provided $\mathfrak{L}_\alpha$ and $\mathfrak{L}_\beta$ are equivalent. Since

$$(3.31) \qquad A_{\alpha\beta} L_\beta = L_\alpha A_{\alpha\beta}$$

implies

$$A_{\alpha'\beta'} L_{\beta'} = L_{\alpha'} A_{\alpha'\beta'},$$

if (α', β') is conjugate to (α, β) this coördination has the property mentioned under (3.1), and is hence being generated by a certain subfield κ of degree v, the *central field*; $n = v \cdot m$. The quotient m is called the *Schur index* of $\mathfrak{L}$. We are able to determine the non-singular $A_{\alpha\beta}$ such that $A_{\alpha\alpha} = E_f$ and such that they form a conjugate double set over κ. In the future, subscripts α or $\alpha\beta$ or $\alpha\beta\gamma$ are always meant to indicate that we are concerned with a conjugate set over κ.

On passing into a field K involving all the conjugate fields $k(\vartheta_\alpha)$ one sees from Burnside's theorem and its supplement (1.4-A) that the order h of $\mathfrak{L}$ equals $v \cdot f^2$. From this follows by means of

$$h = tg = tfn = tfvm$$

that

$$(3.32) \qquad f = tm.$$

The arbitrariness in choosing the $A_{\alpha\beta}$ consists in the possibility of replacing $A_{\alpha\beta}$ by $e_{\alpha\beta} A_{\alpha\beta}$; $e_{\alpha\alpha} = 1$, $e_{\alpha\beta} \neq 0$.

The equivalences $\mathfrak{L}_\alpha \sim \mathfrak{L}_\beta$, $\mathfrak{L}_\beta \sim \mathfrak{L}_\gamma$:

$$A_{\alpha\beta} L_\beta = L_\alpha A_{\alpha\beta}, \qquad A_{\beta\gamma} L_\gamma = L_\beta A_{\beta\gamma}$$

in the following way result in the equivalence $\mathfrak{L}_\alpha \sim \mathfrak{L}_\gamma$:

$$A_{\alpha\beta} A_{\beta\gamma} \cdot L_\gamma = L_\alpha \cdot A_{\alpha\beta} A_{\beta\gamma}.$$

Hence a relation

$$(3.33) \qquad A_{\alpha\beta} A_{\beta\gamma} = c_{\alpha\beta\gamma} A_{\alpha\gamma}$$

must hold. The conjugate numbers $c_{\alpha\beta\gamma} \neq 0$ form the *factor set*. From (3.33) and $A_{\alpha\alpha} = E$ follows at once

$$(3.34) \qquad \begin{cases} c_{\alpha\alpha\beta} = 1, \qquad c_{\alpha\beta\beta} = 1; \\ c_{\alpha\beta\gamma} \cdot c_{\alpha\gamma\delta} = c_{\alpha\beta\delta} \cdot c_{\beta\gamma\delta}. \end{cases}$$

If one replaces $A_{\alpha\beta}$ by $e_{\alpha\beta} A_{\alpha\beta}$ the factor set c is changed into the "equivalent" c^*:

$$c^*_{\alpha\beta\gamma} = \frac{e_{\alpha\beta} e_{\beta\gamma}}{e_{\alpha\gamma}} \cdot c_{\alpha\beta\gamma}.$$

The splitting field once chosen, the factor set is uniquely determined by $\mathfrak{l}$ in the sense of equivalence.

(3.4) Conversely $\mathfrak{L}$ *is uniquely determined by its factor set in the sense of equivalence.* The proof obviously must depend on ascertaining the following two facts: 1) A matrix M,

$$\left\| \begin{matrix} M_1 & & \\ & \ddots & \\ & & M_n \end{matrix} \right\| \sim M, \tag{3.41}$$

whose parts M_α form a conjugate set of matrices satisfying the equations (3.31):

$$A_{\alpha\beta} M_\beta = M_\alpha A_{\alpha\beta} \tag{3.42}$$

necessarily lies in $\mathfrak{L}$. 2) If a second system $A^*_{\alpha\beta}$ satisfies the same equations (3.33) as $A_{\alpha\beta}(A^*_{\alpha\alpha} = E)$, then

$$A^*_{\alpha\beta} = T_\alpha^{-1} A_{\alpha\beta} T_\beta \tag{3.43}$$

where T_α are conjugate non-singular matrices.

Proof of 1). M, if defined according to (3.41) by means of arbitrary matrices M_α in K which fulfill the equations (3.42), contains just the right number $v \cdot f^2$ of parameters and is hence contained in the closure $\mathfrak{L}_K$. If in addition, the M_α are conjugate matrices in $k(\vartheta_\alpha)$, the matrix M itself is rational and hence lies in $\mathfrak{L}$. [By the way, our proposition shows that the matrix L defined by $L_\alpha = \eta_\alpha E$ lies in $\mathfrak{L}$ provided η is any number of the central field; which proves that in the isomorphism $a \to \vartheta$ the central field corresponds to that subfield of $k(a)$ which consists of the centrum elements of $\mathfrak{A}$.]

Proof of 2).

$$T_\alpha = A^{*-1}_{\rho\alpha} A_{\rho\alpha}$$

satisfies the equation (3.43) for every fixed ρ (coördinated with $\alpha, \beta, \cdots$); the only trouble is that this is not a matrix lying in $k(\vartheta_\alpha)$! We therefore form

$$T_\alpha = \sum_\rho \zeta_\rho A^{*-1}_{\rho\alpha} A_{\rho\alpha} \tag{3.44}$$

by means of an arbitrary number ζ of $k(\vartheta)$ and must try to take care that the determinant $| T_\alpha | \neq 0$. When we put (Vandermonde transformation!)

$$\zeta_\rho = z_0 + z_1 \vartheta_\rho + \cdots + z_{n-1} \vartheta_\rho^{n-1}$$

the determinant $| T_1 |$ is a polynomial of the variables $z_0, z_1, \cdots, z_{n-1}$ that does not vanish identically since $| T_1 | = 1$ for $\zeta_1 = 1$, $\zeta_\rho = 0$ $(\rho \neq 1)$. Consequently there exist also *values* z_i *in* k for which $| T_1 | \neq 0$; then the T_α are non-singular conjugate matrices. 2) in particular contains Speiser's theorem: if $A_{\alpha\beta} A_{\beta\gamma} = A_{\alpha\gamma}$, then there exist non-singular conjugate matrices T_α such that $A_{\alpha\beta} = T_\alpha^{-1} T_\beta$.

[The existential question is the following: Given a field $k(\vartheta)$ of degree $n = v \cdot m$ and a subfield $\kappa = k(\eta)$ of degree v; the conjugations $\vartheta \to \vartheta_\alpha$, $\vartheta \to \vartheta_\beta$

are called coördinated if $\eta_\alpha = \eta_\beta$. Furthermore, given a set of numbers $c_{\alpha\beta\gamma} \neq 0$ conjugate over κ and satisfying the relations (3.34): Does there exist a simple algebra for which $k(\vartheta)$, κ, $c_{\alpha\beta\gamma}$ play the part of splitting field, central field, and factor set, respectively? Brauer answers it affirmatively by giving an example;[6] the equations (3.33) have a solution $A_{\alpha\beta}$ *of degree m*. But what one obtains may correspond to the more general situation only cursorily mentioned above, that one failed to choose a splitting field of minimum degree. If one wishes to exclude this, one has to assume in addition that the given factor set is of *Schur index m*, i.e., that the equations (3.33) allow of no solution of lower degree than m.]

(3.5) We need a certain generalization of the theory of splitting fields which I contrast, by the word "*relative*," to the *absolute* splitting fields heretofore studied. The splitting of $\mathfrak{L}$ into the $\mathfrak{L}_\alpha$ may have been accomplished again by an element a of the commutator algebra. We apply the old notations. However, we shall now assume only that the parts $\mathfrak{L}_\alpha$ are irreducible in a given field K including the n conjugate fields $k(\vartheta_\alpha)$. (For the application to Riemann matrices, K will be the "real" field.) That is to say, we rise merely to the level K rather than to "absolute" irreducibility. In following the above procedure we are to consider those elements Q of $\mathfrak{A}$ that commute with A. There occur the parallel decompositions

$$A \text{ in } \vartheta_\alpha E\,, \qquad Q \text{ in } Q_\alpha\,, \qquad L \text{ in } L_\alpha\,.$$

The extension of the linear set $\mathfrak{Q}_\alpha$ of all Q_α to $k(\vartheta_\alpha)$ may be called $\mathfrak{Q}^{(\alpha)}$. As $\mathfrak{L}_\alpha$ is irreducible in $k(\vartheta_\alpha)$, this $\mathfrak{Q}^{(\alpha)}$ is a division algebra of a certain order d in $k(\vartheta_\alpha)$, the abstract scheme of which may be called $\mathfrak{q}^{(\alpha)}$. The laws of composition in the several $\mathfrak{q}^{(\alpha)}$ are conjugate to each other in the fields $k(\vartheta_\alpha)$; they are copies of a model division algebra $\mathfrak{q}$ in $k(\vartheta)$. The element $q^{(\alpha)}$ of $\mathfrak{q}^{(\alpha)}$ is represented in $\mathfrak{Q}^{(\alpha)}$ by the matrix $Q^{(\alpha)} = (q^{(\alpha)}) \times E_f$ where $(q^{(\alpha)})$ denotes the regular representation of $q^{(\alpha)}$ in $\mathfrak{q}^{(\alpha)}$. The former notation is changed to the effect that now $d \cdot f$ is the degree of the matrices $Q^{(\alpha)}$, L_α. The order of $\mathfrak{L}_\alpha$ is $d \cdot f^2$ according to Theorem (1.3-B).

Since $\mathfrak{L}_\alpha$ is irreducible in K, $\mathfrak{q}^{(\alpha)}$ remains a division algebra when we close it in K: $\mathfrak{q}_K^{(\alpha)}$. The elements $q^{(\alpha)}$ of $\mathfrak{q}_K^{(\alpha)}$ shall be called *α-quantics*. The upper index (α) shall always indicate an α-quantic. The situation is now perfectly analogous to the previous one but for the fact that quantics take the place of scalars.

α and β are coördinated provided $\mathfrak{L}_\alpha$ and $\mathfrak{L}_\beta$ are equivalent in K. The coördination is effected by a subfield κ of $k(\vartheta)$, the central field of degree v. According to Theorem (1.4-A) the v non-coördinated parts $\mathfrak{L}_\alpha$ are entirely independent of each other in the closure $\mathfrak{L}_K$. The order of $\mathfrak{L}$ is therefore $h = v \cdot df^2$;

[6] *Math. Zeitschrift* vol. 28 (1928), pp. 677–696, in particular §6, p. 682.—The whole theory of factor sets is due to R. Brauer: *Sitzungsber. Berl. Akad.* (1926), pp. 410–416. Compare furthermore: R. Brauer, *Math. Zeitschrift*, vol. **30** (1929), pp. 79–107.

comparison with the degree $g = mvdf$ again leads to the relation $f = tm$. The equation

$$BL_\beta = L_\alpha B \,, \tag{3.51}$$

when required to hold for all L has only the solution $B = 0$ if α and β are not coördinated. If they are coördinated, however, it has a non-singular solution B, and every solution $Q^{(\alpha)}B = q^{(\alpha)}B$ arises from it by fore multiplication with an α-quantic—or by aft multiplication with a β-quantic:

$$q^{(\alpha)}B = Bq^{(\beta)} \,. \tag{3.52}$$

Any solution different from zero is therefore non-singular. B, by means of (3.52), establishes an isomorphism T: $q^{(\alpha)} \leftrightarrow q^{(\beta)}$ between the α- and the β-quantics. Let us stop for a moment to consider how this isomorphism is changed when one replaces B by $b^{(\alpha)}B(= Bb^{(\beta)})$ ($b^{(\alpha)}$, an α-quantic). The new isomorphism is defined by

$$q^{(\alpha)}b^{(\alpha)}B = b^{(\alpha)}Bq^{(\beta)} \,. \tag{3.53}$$

We form

$$b^{(\alpha)-1}\, q^{(\alpha)}\, b^{(\alpha)} = \bar{q}^{(\alpha)} \,. \tag{3.54}$$

Then (3.53) reads:

$$\bar{q}^{(\alpha)}B = Bq^{(\beta)} \,,$$

and consequently $\bar{q}^{(\alpha)} \rightarrow q^{(\beta)}$ is the old isomorphism T. The modification consists in letting T be preceded by the inner automorphism (3.54), $q^{(\alpha)} \rightarrow \bar{q}^{(\alpha)}$, of the α-quantics generated by $b^{(\alpha)}$ (or in having the inner automorphism $[b^{(\beta)}]$ of the β-quantics follow T). The inner automorphism generated by an element b is briefly denoted by $[b]$.

It is perhaps advisable to describe our "quantics" a little more carefully. Each quantic x is given as a set of d numbers $(x_1, \cdots, x_d)$ in K; the coefficients π in the multiplication law

$$xy = z\colon\ z_i = \sum_{k,l} \pi^{ikl} x_k y_l \qquad (i, k, l = 1, \cdots, d)$$

are numbers in $k(\vartheta)$. Transition to a new base is described by equations

$$x_i = \sum_k \tau^{ik} \bar{x}_k \qquad (|\,\tau^{ik}\,| \neq 0)$$

with coefficients τ^{ik} in $k(\vartheta)$; only such relations are to be studied as are invariant under arbitrary changes in base of this type. We manufacture n copies $\mathfrak{q}^{(\alpha)}$ of this model $\mathfrak{q}$ (α-quantics, $\alpha = 1, \cdots, n$) by replacing the π^{ikl} by their conjugates π_α^{ikl} in $k(\vartheta_\alpha)$. A change of base takes place simultaneously in all n copies, the coefficients τ^{ik} being replaced by the conjugates τ_α^{ik} in the α^{th} copy (think of the show girls again!). It has an invariantive meaning to say that an α-quantic $x^{(\alpha)} = (x_1^{(\alpha)}, \cdots, x_d^{(\alpha)})$ lies, let us say, in $k(\vartheta_\alpha, \vartheta_\beta)$: $x_i^{(\alpha)}$ in $k(\vartheta_\alpha, \vartheta_\beta)$; and it has

an invariantive meaning to assert that a set $x^{(\alpha)}_{\alpha\beta}$ of quantics are conjugate (over κ). It has an invariantive meaning to state that a given isomorphism T between the α- and β-quantics:

$$x^{(\alpha)} \leftrightarrow x^{(\beta)}:\ \bar{x}^{(\beta)}_i = \sum_j \sigma_{ij} x^{(\alpha)}_j$$

lies in $k(\vartheta_\alpha, \vartheta_\beta)$: σ_{ij} in $k(\vartheta_\alpha, \vartheta_\beta)$; and that a set $T_{\alpha\beta}$ of such isomorphisms is conjugate over κ.

If α and β are coördinated, (3.51) has a solution $B = A_{\alpha\beta} \neq 0$ in $k(\vartheta_\alpha, \vartheta_\beta)$; it is non-singular. We take care that $A_{\alpha\alpha} = E$ and $A_{\alpha\beta}$ form, as their notation indicates, a conjugate set over κ. By means of the formula

$$q^{(\alpha)} A_{\alpha\beta} = A_{\alpha\beta} q^{(\beta)} \tag{3.55}$$

$A_{\alpha\beta}$ determines an isomorphism $T_{\alpha\beta}$: $q^{(\alpha)} \leftrightarrow q^{(\beta)}$ between the α- and the β-quantics; again, the $T_{\alpha\beta}$ are conjugate over κ. We must have an equation

$$A_{\alpha\beta} A_{\beta\gamma} = c^{(\alpha)}_{\alpha\beta\gamma} A_{\alpha\gamma} \left(= A_{\alpha\gamma} c^{(\gamma)}_{\alpha\beta\gamma}\right), \tag{3.56}$$

the c's being a triple set of conjugate quantics $\neq 0$. This equation proves that the succession of the two isomorphisms

$$T_{\alpha\beta}:\ q^{(\alpha)} \to q^{(\beta)}, \qquad T_{\beta\gamma}:\ q^{(\beta)} \to q^{(\gamma)}$$

results in an isomorphism between the α- and γ-quantics, equal to $T_{\alpha\gamma}$ preceded by the inner automorphism $[c^{(\alpha)}_{\alpha\beta\gamma}]$. When we remember our convention that subscripts α or $\alpha\beta$ or $\alpha\beta\gamma$ shall automatically indicate that the terms are conjugate over κ, and that an upper index (α) designates an α-quantic, we may finally describe *a quantic factor set* as follows:

Given a field $k(\vartheta)$ of degree n over k; a subfield κ of degree v, $n = v \cdot m$, determines the coördinating of the conjugations $\vartheta \to \vartheta_\alpha$ into v classes; a field K encompasses all conjugate fields $k(\vartheta_\alpha)$.

Given a division algebra $\mathfrak{q}$ of "quantics" in K of the nature above described: the multiplication law has coefficients π in $k(\vartheta)$ and only base transformations with coefficients τ in $k(\vartheta)$ are allowed. We then have the n conjugate copies $\mathfrak{q}^{(\alpha)}$ of the model $\mathfrak{q}$: α-quantics.

A factor set consists: 1) *of a κ-conjugate set of isomorphisms $T_{\alpha\beta}$: $q^{(\alpha)} \leftrightarrow q^{(\beta)}$, and* 2) *a κ-conjugate set of quantics $\neq 0$:*

$$c^{(\alpha)}_{\alpha\beta\gamma} \leftrightarrow c^{(\gamma)}_{\alpha\beta\gamma} \tag{3.57}$$

such that the succession of $T_{\alpha\beta}$ and $T_{\beta\gamma}$ results in $T_{\alpha\gamma}$ preceded by the inner automorphism $[c^{(\alpha)}_{\alpha\beta\gamma}]$ (or succeeded by the inner automorphism $[c^{(\gamma)}_{\alpha\beta\gamma}]$). The following conditions prevail:

$$\begin{cases} c^{(\alpha)}_{\alpha\alpha\beta} = 1, & c^{(\beta)}_{\alpha\beta\beta} = 1, \\ c^{(\alpha)}_{\alpha\beta\gamma} \cdot c^{(\alpha)}_{\alpha\gamma\delta} \leftrightarrow c^{(\delta)}_{\alpha\beta\delta} \cdot c^{(\delta)}_{\beta\gamma\delta}. \end{cases} \tag{3.58}$$

In analogy to proposition 1) in (3.4), we have the

LEMMA (3.5): A matrix M in k breaking up into parts M_α will lie in $\mathfrak{L}$ provided M_α commutes with all the $Q^{(\alpha)}$ and the relations

$$A_{\alpha\beta}M_\beta = M_\alpha A_{\alpha\beta}$$

are satisfied for each pair of coördinated indices α, β.

The proof is the same as before: these conditions reduce the number of parameters in M to the right value $v \cdot df^2$.

[Ascent from our present level K to the absolute is accomplished by means of a "maximum" element q of $\mathfrak{q}$ lying in $k(\vartheta)$; it cracks each $\mathfrak{L}_\alpha$ into absolutely irreducible parts according to the numerically distinct roots of q.

Of particular interest is the special case that our quantics are commutative. Then we have

$$T_{\beta\gamma}T_{\alpha\beta} = T_{\alpha\gamma} ,$$

hence by Speiser's theorem: $T_{\alpha\beta} = T_\beta T_\alpha^{-1}$. This means: there exists a base for $\mathfrak{q}$ in terms of which the multiplication law has coefficients in κ. $\mathfrak{q}$ may then be described as a commutative field over κ that is not reduced by the extension of the reference field κ to K.]

4. Splitting Field of a Riemann Algebra

(4.1) Now let $\mathfrak{L}$ be again the irreducible algebra of matrices in k associated with a pure Riemann matrix $R = C^{-1}S$ and $\mathfrak{A}$ its commutator algebra. In $\mathfrak{L}$ and $\mathfrak{A}$ we have the anti-automorphic involutions $L \rightarrow L_*$, $A \rightarrow A_*$ generated by C.

LEMMA (4.1) (Rosati). *If A is an even element of the commutator algebra, its roots are real and C and S break up like $\mathfrak{L}$ into parts C_α, S_α according to the numerically distinct roots. The roots of an odd A are pure imaginary.*

For the proof of this lemma it is convenient to operate in the algebraically closed field $(K, \sqrt{-1})$ and to transform C and S in the manner

$$C \rightarrow \bar{U}'CU , \qquad S \rightarrow \bar{U}'SU \qquad (L \rightarrow U^{-1}LU)$$

by means of the transformation U carrying A into its diagonal form. The equation $CR = S$ is preserved and after the transformation, S is the coefficient matrix of a positive definite Hermitian form:

$$\bar{S}' = S , \qquad \bar{C}' = \pm C .$$

A_* is now defined by $\bar{A}_*' = CAC^{-1}$ and thus our even A satisfies the equation

$$\bar{A}'C = CA . \tag{4.11}$$

We broke A into parts $\vartheta_\alpha E$ where ϑ_α are the numerically distinct roots of A $(\alpha = 1, \cdots, n)$. The matrix C is accordingly checkered into squares $C_{\alpha\beta}$, and (4.11) reduces to

$$(\bar{\vartheta}_\alpha - \vartheta_\beta)C_{\alpha\beta} = 0 . \tag{4.12}$$

On account of $CR = S$:

$$C_{\alpha\beta} R_\beta = S_{\alpha\beta} \,,$$

$C_{\alpha\alpha}$ is non-singular and hence (4.12) requires:

$$\bar{\vartheta}_\alpha = \vartheta_\alpha \,, \qquad C_{\alpha\beta} = 0 \quad (\text{for } \alpha \neq \beta) \,.$$

Since the roots ϑ_α are real the corresponding Vandermonde transformation U is also real.

The case of an odd A is treated along the same lines.

(4.2) We now proceed in the same manner as in (3.2), with the difference, however, that only *even* elements a of $\mathfrak{A}$ shall be used for the purpose of splitting. As long as it is still possible to find even elements b commuting with a and not included in the field $k(a)$, one goes on adjoining them until one finds an even a such that every even b commuting with a lies in the field $k(a)$; by this a we determine our *splitting field* $k(\vartheta)$. Rosati's lemma tells us that all the conjugate ϑ_α are real, or that ϑ is a "totally real algebraic number" over k. Stopping here has the disadvantage that we do not get an "absolute" splitting field; the situation is rather that described in (3.5) with the real field K as the level reached. Indeed, the elements $q = Q$ of $\mathfrak{A}$ commuting with our maximal even a form a division algebra over $k(a)$ in which every element q satisfies a quadratic equation in $k(a)$. For if q commutes with a, so does q_*, and $q + q_*$ and qq_* are even and commute with a; they therefore lie in $k(a)$. The relation

$$q^2 - q(q + q_*) + qq_* = 0$$

is obvious. Now the only division algebras over a field $k(\vartheta)$ in which each element satisfies a quadratic equation in the reference field are of the following three types:[7]

I. the "*scalar*": elements q = numbers q_0 in the reference field $k(\vartheta)$;

II. the "*square root*": elements are of form $q_0 + q_1\iota$ where $\iota^2 = -\lambda$; q_0, q_1 vary in $k(\vartheta)$, $-\lambda$ lies, and is not square, in $k(\vartheta)$;

III. the "*quaternion*": elements are of form

$$q_0 + q_1\iota_1 + q_2\iota_2 + q_3\iota_3$$

where

$$\iota_1\iota_2 = -\iota_2\iota_1 = \iota_3 \,, \qquad \iota_1^2 = -\lambda \,, \qquad \iota_2^2 = -\mu$$

and q_0, q_1, q_2, q_3 vary in the reference field while $-\lambda$ and $-\mu$ lie, and are no squares, in $k(\vartheta)$.

Thus one of these three types plays the rôle of our algebra $\mathfrak{q}$ of "quantics." The Rosati lemma, however, provides some more information. In the case of II and III the elements q represented by ι or ι_1, ι_2 satisfy an irreducible *pure*

[7] Cf. for example: L. E. Dickson, *Algebren und ihre Zahlentheorie*, Zürich (1927), pp. 43–45.

quadratic equation in $k(\vartheta)$, and hence $q + q_* = 0$, or q is odd. Therefore its roots must be pure imaginary, or λ in case II and λ, μ in case III are *totally positive* (all the conjugates λ_α; λ_α, μ_α respectively, are positive). For this reason we call the square root II and the quaternion III "*totally negative.*" In consequence thereof, *each of the algebras* I, II, III *in all their n conjugate "copies"* $\mathfrak{q}^{(\alpha)}$ *remains a division algebra when extended to the real field K.* We have the parallel decompositions of

$$A \text{ into } \vartheta_\alpha E\ , \qquad Q \text{ into } Q_\alpha\ , \qquad L \text{ into } L_\alpha$$
$$[R \text{ into } R_\alpha\ , \qquad C \text{ into } C_\alpha\ , \qquad S \text{ into } S_\alpha]\ .$$

$\mathfrak{L}_\alpha$ is irreducible in $k(\vartheta_\alpha)$. Our remark proves that the commutator algebra $\mathfrak{Q}^{(\alpha)}$ of $\mathfrak{L}_\alpha$ in $k(\vartheta_\alpha)$ remains a division algebra *under extension to K* from which fact the criterion (1.3-D) permits drawing the inference that $\mathfrak{L}_\alpha$ *is irreducible in K.* Furthermore we should keep in mind that in $\mathfrak{Q}^{(\alpha)}$ our involution $q^{(\alpha)} \to q_*^{(\alpha)}$ consists in the transition from a quantic q to its "complex conjugate" q_* defined by:

$$(4.21) \qquad \begin{array}{l|l|l} q\ = q_0 & q\ = q_0 + q_1\iota & q\ = q_0 + q_1\iota_1 + q_2\iota_2 + q_3\iota_1 \\ q_* = q_0 & q_* = q_0 - q_1\iota & q_* = q_0 - q_1\iota_1 - q_2\iota_2 - q_3\iota_3 \end{array}$$

respectively. In each case we have

$$Q^{(\alpha)} = (q^{(\alpha)}) \times E_f\ .$$

Main Theorem, First Part. *A Riemann algebra* $\mathfrak{L}$ *splits over a certain totally real field* $k(\vartheta)$ *of degree* $n = mv$ *with its central field* κ *of degree v into parts of degree df which are irreducible in the real field K. It is described relatively to* $k(\vartheta)$ *by a quantic factor set where the algebra of quantics of order d is either scalar* ($d = 1$) *or a totally negative square root field* ($d = 2$), *or a totally negative quaternion* ($d = 4$).

[In cases II and III ascent to an absolute splitting field would be accomplished by adjoining the square root $\sqrt{-\lambda}$; we prefer, however, to stop at the totally real field $k(\vartheta)$.[8]

(4.3) Before going on we shall mention a few elementary features common to our three algebras $\mathfrak{q}_K$ of quantics q. The product of a q with its complex-conjugate q_*, (4.21), is a positive scalar $N(q)$, the *norm* of q:

$$N(q) = q_0^2 \quad | \quad q_0^2 + \lambda q_1^2 \quad | \quad q_0^2 + \lambda q_1^2 + \mu q_2^2 + \lambda\mu q_3^2,$$

which satisfies the multiplicative law:

$$N(pq) = N(p) \cdot N(q)\ .$$

[8] Albert adjoins to his Galois splitting field $k(\vartheta_1, \cdots, \vartheta_n)$ the extraneous real square roots $\sqrt{\lambda_\alpha}$, $\sqrt{\mu_\alpha}$ in order to make the case III more easily accessible; here we want to avoid the introduction of such irrationalities foreign to the problem.

After our algebra $\mathfrak{q}$ has been closed in K one may choose as "units"

$$\iota/\sqrt{\lambda} = i \quad | \quad \iota_1/\sqrt{\lambda} = i_1, \quad \iota_2/\sqrt{\mu} = i_2$$

in cases II and III; one then has to deal with the Gauss field $K(i)$ and the ordinary Hamilton quaternions, respectively. An automorphism $q \to p$ in the latter case is expressed in terms of the units $i_1, i_2, i_3 = i_1 i_2$ by equations:

$$\begin{aligned} i_1 &\to a + a_1 i_1 + a_2 i_2 + a_3 i_3 = j_1, \\ i_2 &\to b + b_1 i_1 + b_2 i_2 + b_3 i_3 = j_2, \qquad \text{(all } a, b, c \text{ real numbers)} \\ i_3 &\to c + c_1 i_1 + c_2 i_2 + c_3 i_3 = j_3 . \end{aligned}$$

The requirement $j_1^2 = -1$ yields:

$$a^2 - a_1^2 - a_2^2 - a_3^2 = -1 ; \qquad 2aa_1 = 2aa_2 = 2aa_3 = 0 .$$

Since simultaneous vanishing of a_1, a_2, a_3 would contradict the first equation, we must have $2a = 0$ and for the same reason $2b = 0$, $2c = 0$. This means that the automorphism $q \to p$ carries q_* into p_*. Consequently the norm $N(q)$ is left invariant. Adding the simpler cases I and II we may state our result in the following

LEMMA (4.3-A). *An isomorphism T between α- and β-quantics matches $q_*^{(\alpha)} \leftrightarrow q_*^{(\beta)}$ as well as $q^{(\alpha)} \leftrightarrow q^{(\beta)}$. It leaves the norm invariant*: $N_\alpha(q^{(\alpha)}) = N_\beta(q^{(\beta)})$.

In computing explicitly the multiplication $(q) : x' = qx$ in our quaternion algebra, one finds

$$(4.31) \qquad (q) = \begin{Vmatrix} q_0, & -\lambda q_1, & -\mu q_2, & -\lambda\mu q_3 \\ q_1, & q_0, & -\mu q_3, & \mu q_2 \\ q_2, & \lambda q_3, & q_0, & -\lambda q_1 \\ q_3, & -q_2, & q_1, & q_0 \end{Vmatrix}$$

and we verify the relation

$$(q_*)' = (n)(q)(n)^{-1}$$

where

$$(4.32) \qquad (n) = \begin{Vmatrix} 1 & & & \\ & \lambda & & \\ & & \mu & \\ & & & \lambda\mu \end{Vmatrix}$$

is the coefficient matrix of the norm. It is important to observe that (n) is symmetric and positive-definite. Adding again the simpler cases I and II we thus proved

LEMMA (4.3-B):

$$(q_*)' = (n)(q)(n)^{-1} \tag{4.33}$$

where (n) *is the coefficient matrix of the norm.*]

5. The Norm Condition

(5.1) Before attacking the slightly more difficult cases II and III we treat, as a model, case I where our totally real splitting field $k(\vartheta)$ splits $\mathfrak{L}$ into *absolutely* irreducible parts $\mathfrak{L}_\alpha$.

The equation

$$L'(l_*)C = CL(l)$$

defining the involution $l \to l_*$ of $\mathfrak{L}$ splits into the relations

$$L'_\alpha(l_*) = C_\alpha L_\alpha(l) C_\alpha^{-1} . \tag{5.11}$$

We have

$$L_\alpha A_{\alpha\beta} = A_{\alpha\beta} L_\beta . \tag{5.12}$$

The $\check{A}_{\alpha\beta} = A'^{-1}_{\alpha\beta}$ fulfill the same conditions with respect to the L'_α as the $A_{\alpha\beta}$ themselves relatively to the L_α and $C_\alpha A_{\alpha\beta} C_\beta^{-1}$ relatively to $C_\alpha L_\alpha C_\alpha^{-1}$. Hence (5.11) leads to a relation of the form

$$\check{A}_{\alpha\beta} = e_{\alpha\beta} \cdot C_\alpha A_{\alpha\beta} C_\beta^{-1} \tag{5.13}$$

with a conjugate set of numbers $e_{\alpha\beta}$, or

$$C_\beta = e_{\alpha\beta} \cdot A'_{\alpha\beta} C_\alpha A_{\alpha\beta} . \tag{5.14}$$

When we perform the transition from C_α to C_γ on the one hand directly in accordance with this equation, and on the other hand by passing through C_β, we find in making use, for the second process, of the equation

$$A_{\alpha\beta} A_{\beta\gamma} = c_{\alpha\beta\gamma} A_{\alpha\gamma}$$

that the skew-symmetric form C_γ is the transform $A'_{\alpha\gamma} C_\alpha A_{\alpha\gamma}$ of C_α multiplied by $e_{\alpha\gamma}$ on the one hand, or by $e_{\alpha\beta} e_{\beta\gamma} \cdot c^2_{\alpha\beta\gamma}$ on the other hand. Hence

$$c^2_{\alpha\beta\gamma} = e_{\alpha\gamma}/e_{\alpha\beta} e_{\beta\gamma} \ (\sim 1) . \tag{5.15}$$

The numbers $e_{\alpha\beta}$ must be positive as is shown by the following simple observation. The $P_\alpha = C_\alpha L_\alpha$, because of (5.12), satisfy the same relation (5.14) as C_α:

$$P_\beta = e_{\alpha\beta} \cdot A'_{\alpha\beta} P_\alpha A_{\alpha\beta} \tag{5.16}$$

if L lies in $\mathfrak{L}$ or its closure $\mathfrak{L}_K$. Since $CR = S$, $C_\alpha R_\alpha = S_\alpha$ we have in particular

$$S_\beta = e_{\alpha\beta} \cdot A'_{\alpha\beta} S_\alpha A_{\alpha\beta} . \tag{5.17}$$

All quadratic forms S_α are positive definite. By the transformation $A_{\alpha\beta}$ the positive form S_α is carried into the positive form $A'_{\alpha\beta}S_\alpha A_{\alpha\beta}$. By (5.17) this coincides with the positive form S_β but for the factor $e_{\alpha\beta}$; hence this factor is to be positive. We have arrived at the following result:

The factor set $c_{\alpha\beta\gamma}$ of a Riemann algebra $\mathfrak{L}$ of type I *satisfies relations*

$$c^2_{\alpha\beta\gamma} = e_{\alpha\gamma}/e_{\alpha\beta}e_{\beta\gamma} \tag{5.15}$$

where $e_{\alpha\beta}$ is a double set of positive numbers conjugate over the central field κ. We say that c^2 is *totally positive equivalent* 1.

The condition is not only necessary but sufficient. *For let* (5.15) *be fulfilled.* These equations state that $\breve{A}_{\alpha\beta}$ has the same factor set as $e_{\alpha\beta}A_{\alpha\beta}$, and we know by proposition 2) in (3.4) that from this an equivalence like (5.13) follows with some conjugate non-singular matrices C_α. We constructed such a C_α, cf. (3.44), by means of the formula

$$C_\alpha = \sum_\rho \zeta_\rho e_{\rho\alpha}\breve{A}^{-1}_{\rho\alpha}A_{\rho\alpha} = \sum_\rho \zeta_\rho e_{\rho\alpha}A'_{\rho\alpha}A_{\rho\alpha}$$

where ζ is a number in $k(\vartheta)$. Let us take ζ in particular as a square number $\zeta = \xi^2$, ξ in $k(\vartheta)$, so that ζ is totally positive. The coefficients $e_{\rho\alpha}$ are positive by assumption. $A'A$ is a positive symmetric matrix if A is real; it is indeed the transform of the unit matrix E by the transformation A. *Hence our*

$$C_\alpha = \sum_\rho e_{\rho\alpha}\zeta_\rho A'_{\rho\alpha}A_{\rho\alpha} \qquad (\zeta_\rho > 0) \tag{5.18}$$

is symmetric, positive, and therefore non-singular; no special precautions against possible degeneration are necessary. This is the essential part of the proof of sufficiency; it needs some elementary supplement which the last section will take care of, but the pivot of our whole demonstration consists in the two formulas (5.17), (5.18), the first proving (5.15) and $e_{\alpha\beta} > 0$ to be a necessary condition, the second warranting the existence of a symmetric positive C, once this condition is fulfilled.

(5.2) It is easy to survey the *modifications needed to adapt our considerations to the other cases* II *and* III. As a consequence of

$$A_{\alpha\beta}L_\beta = L_\alpha A_{\alpha\beta}$$

we have

$$\breve{A}_{\alpha\beta}L'_\beta(l) = L'_\alpha(l)\breve{A}_{\alpha\beta}.$$

Hence $C_\alpha^{-1}\breve{A}_{\alpha\beta}C_\beta$ have the same significance for $C_\alpha^{-1}L'_\alpha(l)C_\alpha = L_\alpha(l_*)$ and therefore

$$C_\alpha^{-1}\breve{A}_{\alpha\beta}C_\beta = Q^{(\alpha)}A_{\alpha\beta}(= q^{(\alpha)}A_{\alpha\beta}) \qquad [Q^{(\alpha)} \text{ in } \mathfrak{Q}^{(\alpha)}]$$

or

$$C_\beta = A'_{\alpha\beta}C_\alpha Q^{(\alpha)}A_{\alpha\beta}. \tag{5.21}$$

We must try to prove that $q^{(\alpha)}$ is a scalar. Putting the $'$ on the whole equation (5.21) we get because of $C'_\alpha = \pm C_\alpha$:

$$C_\beta = A'_{\alpha\beta} Q^{(\alpha)\prime} C_\alpha A_{\alpha\beta}$$

which changes by

$$C_\alpha^{-1} Q^{(\alpha)\prime} C_\alpha = Q_*^{(\alpha)} \tag{5.22}$$

into

$$C_\beta = A'_{\alpha\beta} C_\alpha Q_*^{(\alpha)} A_{\alpha\beta}.$$

Comparison with (5.21) shows that $q_*^{(\alpha)} = q^{(\alpha)}$, and hence $q^{(\alpha)}$ is a scalar. We denote it by $e_{\alpha\beta}$ as before, and then obtain the equations (5.14), (5.17) with their implication $e_{\alpha\beta} > 0$.

Let us write the equation

$$A_{\alpha\beta} A_{\beta\gamma} = c^{(\alpha)}_{\alpha\beta\gamma} A_{\alpha\gamma}$$

in which $c^{(\alpha)}_{\alpha\beta\gamma}$ stands for the matrix $Q^{(\alpha)}$ corresponding to the α-quantic $q^{(\alpha)} = c^{(\alpha)}_{\alpha\beta\gamma}$ in the form

$$A_{\alpha\beta} A_{\beta\gamma} = Q^{(\alpha)} A_{\alpha\gamma}.$$

If we now proceed as before we find on the one hand

$$C_\gamma = e_{\alpha\gamma} \cdot A'_{\alpha\gamma} C_\alpha A_{\alpha\gamma}$$

and on the other

$$C_\gamma = e_{\alpha\beta} e_{\beta\gamma} A'_{\alpha\gamma} Q^{(\alpha)\prime} C_\alpha Q^{(\alpha)} A_{\alpha\gamma}.$$

Making use again of (5.22) the middle factors

$$Q^{(\alpha)\prime} C_\alpha Q^{(\alpha)} \text{ change into } C_\alpha Q_*^{(\alpha)} Q^{(\alpha)} = N(q^{(\alpha)}) C_\alpha,$$

and our result is

$$N(q^{(\alpha)}) = \frac{e_{\alpha\gamma}}{e_{\alpha\beta} e_{\beta\gamma}}.$$

For its full appreciation one should observe that (3.57), (3.58) by the multiplicative property of the norm and lemma (4.3-A) imply

$$N(c^{(\alpha)}_{\alpha\beta\gamma}) = N(c^{(\gamma)}_{\alpha\beta\gamma});$$
$$N(c^{(\alpha)}_{\alpha\beta\gamma}) \cdot N(c^{(\alpha)}_{\alpha\gamma\delta}) = N(c^{(\delta)}_{\alpha\beta\delta}) \cdot N(c^{(\delta)}_{\beta\gamma\delta}).$$

This means that *the norm of a quantic factor set of types* I, II *or* III *is a scalar factor set.* We have ascertained the following necessary condition, that the algebra described by a quantic factor set of this type relatively to $k(\vartheta)$, be a Riemann algebra:

The norm of the factor set must be totally positive equivalent 1.

(5.3) Vice versa, *if this condition prevails* we proceed to the construction of a pure Riemann matrix associated with $\mathfrak{L}$ in the following way. The first step is to define an anti-automorphic involution in $\mathfrak{L}$ by an appropriately chosen C. From the matrix (n) mentioned in Lemma (4.3-B) and its conjugates (n_α) we form $(n_\alpha) \times E_f = N_\alpha$ and then put

$$C_\alpha = \sum_\rho e_{\rho\alpha} \zeta_\rho A'_{\rho\alpha} N_\rho A_{\rho\alpha}. \tag{5.31}$$

ζ is a totally positive number in $k(\vartheta)$, $\zeta_\rho > 0$; for instance, we choose $\zeta = \xi^2$, ξ in $k(\vartheta)$. $e_{\rho\alpha}$ is >0, and $A'_{\rho\alpha} N_\rho A_{\rho\alpha}$ is the symmetric positive N_ρ transformed by $A_{\rho\alpha}$; each term of our sum (5.31) and consequently the whole sum is symmetric and positive. C is the rational matrix that breaks up into the conjugate C_α. We are going to prove that C_α fulfills the conditions required:

$$Q^{(\alpha)\prime} C_\alpha = C_\alpha Q_*^{(\alpha)}, \tag{5.32}$$

$$C_\beta = e_{\alpha\beta} \cdot A'_{\alpha\beta} C_\alpha A_{\alpha\beta}. \tag{5.33}$$

The relation

$$Q^{(\rho)} A_{\rho\alpha} = A_{\rho\alpha} Q^{(\alpha)} \tag{5.34}$$

defines the isomorphism $T_{\rho\alpha}$: $q^{(\rho)} \leftrightarrow q^{(\alpha)}$. By Lemma (4.3-A), one has at the same time

$$Q_*^{(\rho)} A_{\rho\alpha} = A_{\rho\alpha} Q_*^{(\alpha)},$$

while (5.34) yields

$$A'_{\rho\alpha} Q^{(\rho)\prime} = Q^{(\alpha)\prime} A'_{\rho\alpha}.$$

Therefore

$$C_\alpha Q_*^{(\alpha)} = \sum_\rho e_{\rho\alpha} \zeta_\rho A'_{\rho\alpha} N_\rho Q_*^{(\rho)} A_{\rho\alpha},$$

$$Q^{(\alpha)\prime} C_\alpha = \sum_\rho e_{\rho\alpha} \zeta_\rho A'_{\rho\alpha} Q^{(\rho)\prime} N_\rho A_{\rho\alpha};$$

their coincidence results from the equation (4.33) or

$$Q^{(\rho)\prime} = N_\rho Q_*^{(\rho)} N_\rho^{-1}. \tag{5.35}$$

The right side of (5.33) is by definition

$$\sum_\rho e_{\rho\alpha} e_{\alpha\beta} \zeta_\rho (A_{\rho\alpha} A_{\alpha\beta})' N_\rho (A_{\rho\alpha} A_{\alpha\beta}). \tag{5.36}$$

After putting again

$$c_{\rho\alpha\beta}^{(\rho)} = q^{(\rho)}$$

we have

$$A_{\rho\alpha} A_{\alpha\beta} = Q^{(\rho)} A_{\rho\beta}.$$

This changes the matrix under the sum at the right side of (5.36) into

$$A'_{\rho\beta}Q^{(\rho)\prime}N_\rho Q^{(\rho)}A_{\rho\beta}.$$

According to (5.35),

$$Q^{(\rho)\prime}N_\rho Q^{(\rho)} = N_\rho Q_*^{(\rho)}Q^{(\rho)} = N(q^{(\rho)})\cdot N_\rho.$$

By assumption

$$e_{\rho\alpha}e_{\alpha\beta}N(q^{(\rho)}) = e_{\rho\beta},$$

and thus (5.33) has been verified.

6. Main Theorem

(6.1) The $C_\alpha = C(\vartheta_\alpha)$, as constructed in the last section are the conjugate parts of a rational C. By means of

$$L'(l_*) = CL(l)C^{-1} \qquad L'_\alpha(l_*) = C_\alpha L_\alpha(l)C_\alpha^{-1}$$

it defines an anti-automorphic involution $l \to l_*$ in $\mathfrak{l}$. Indeed, owing to (5.32) and (5.33), the parts M_α of the rational matrix M defined by

$$M' = CLC^{-1}, \qquad M'_\alpha = C_\alpha L_\alpha C_\alpha^{-1}$$

satisfy the relations

$$M_\alpha A_{\alpha\beta} = A_{\alpha\beta}M_\beta$$

and commute with all $Q^{(\alpha)}$ (or $Q_*^{(\alpha)}$) as well as the L_α. M therefore lies in $\mathfrak{L}$, according to Lemma (3.5). $l \to l_*$ is involutorial because of the symmetry of C.

Our C may now be called C_0 and we write $C_0(\vartheta)$ instead of $C(\vartheta)$. The terms *even* and *odd* refer to the involution $l \to l_*$ generated by C_0. Let $\mathfrak{L}^+(\mathfrak{L}^-)$ be the linear k-set of even (odd) elements in $\mathfrak{L}$. The even elements in the extension $\mathfrak{L}_K$ form the extension $\mathfrak{L}_K^+$ of $\mathfrak{L}^+$ to K. Indeed L being an even element in $\mathfrak{L}_K$:

(6.11) $$L = \sum_i z_i L^{(i)}$$

(z_i real numbers, $L^{(i)}$ a base of $\mathfrak{L}$) we obtain by addition of the starred equation to (6.11):

$$2L = \sum_i z_i(L^{(i)} + L_*^{(i)}).$$

The same remark applies to the odd elements of $\mathfrak{L}$ and $\mathfrak{L}_K$. *Let*

$$L^{(i)} \ (i = 1, 2, \cdots, \nu)$$

now be a base of $\mathfrak{L}^+$, and in particular $L^{(1)} = E$.

If we chose C_0 as our C we would obtain *even* Riemann matrices alone. We therefore put

$$C = C_0L_0, \qquad S = C_0L[z]$$

where L_0 is an even or odd non-singular matrix in $\mathfrak{L}$ and $L[z]$ lies in $\mathfrak{L}_K^+$:

$$L[z] = z_1 L^{(1)} + z_2 L^{(2)} + \cdots + z_\nu L^{(\nu)}.$$

$$R = C^{-1}S = L_0^{-1}L[z]$$

shall be our Riemann matrix. We first choose the real numbers z_i so that there exists no homogeneous linear relation among them with rational coefficients.[9] We may normalize $z_1 = 1$. The equation

$$S = C_0 + z_2 \cdot C_0 L^{(2)} + \cdots$$

shows S to be positive definite provided $z_2, \cdots, z_\nu$ are sufficiently small. This can be taken care of[9] without violating the linear independence of the z_i in k by multiplying $z_2, \cdots, z_\nu$ with a common, sufficiently-small, rational factor $\neq 0$.

After the positive character of S is secured, the next question is about the rational commutators A of R. Such a matrix A must commute with all elements $L_0^{-1}L^{(i)}$ which form the base of $L_0^{-1}\mathfrak{L}^+ = \Lambda$; therefore in particular with $L_0^{-1}L^{(1)} = L_0^{-1}$ and consequently with $L^{(i)}$ and with L_0. So we must try to prove the

LEMMA (6.1): *A matrix A commuting with L_0 and the even elements of $\mathfrak{L}$ (or $\mathfrak{L}_K$) commutes with all elements of $\mathfrak{L}$ (or $\mathfrak{L}_K$).*

$L_0^{-1}\mathfrak{L}^+ = \Lambda$ is obviously the linear k-set associated with our R. If L^+ is in $\mathfrak{L}^+$ so is $L_0^{-1}L^+L_0$; hence $\Lambda = \mathfrak{L}^+L_0^{-1}$, and the involution $L \rightarrow L_*$ carries Λ into itself. Λ contains $L_0' = L_0^{-1}L_0^2$; the algebraic closure of Λ—which is the associated algebra of R—thus embraces $\mathfrak{L}^+$ and is either the algebraic closure $(\mathfrak{L}^+)$ of $\mathfrak{L}^+$ or [if L_0^{-1} is not in $(\mathfrak{L}^+)$] the sum of $(\mathfrak{L}^+)$ and $L_0^{-1}(\mathfrak{L}^+)$.

The lemma (6.1) once established, we may be sure that Λ and hence R are rationally *irreducible.* Because Λ is invariant with respect to the involution and C_0 is positive definite, *reduction* of Λ would result in rational *decomposition* according to the proof of Theorem (2.2-B). The matrix, equal to the unit in the one and to zero in the other partial space, would then be a commutator of Λ without being a commutator of $\mathfrak{L}$; for a non-vanishing commutator of $\mathfrak{L}$ is non-singular. The algebraic closure $(\mathfrak{L}^+)$ or $(\mathfrak{L}^+) + L_0^{-1}(\mathfrak{L}^+)$ of Λ must coincide with $\mathfrak{L}$.

(6.2) We split by means of our totally real $k(\vartheta)$ and afterwards extend the individual $\mathfrak{L}_\alpha = \mathfrak{L}(\vartheta_\alpha)$ to K. To prove the lemma (6.1) we must show two things:

1) A matrix $A_{\alpha\beta}$ satisfying the relation

$$A_{\alpha\beta}L_\beta^+ = L_\alpha^+ A_{\alpha\beta}$$

for all L^+ in $\mathfrak{L}^+$ (or $\mathfrak{L}_K^+$) must needs be zero provided α and β are not coördinated.

2) A real matrix A_α commuting with the element $L_0(\vartheta_\alpha)$ and the even $L_\alpha^+ = L^+(\vartheta_\alpha)$ in $\mathfrak{L}_\alpha$ commutes with all L_α.

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[9] When one analyzes the assumptions as to the relation between K and k on which this simple construction depends, one finds this: the ring of all numbers in K that are dominated by k is to form a linear k-set *of infinite order* (or at least of order $\geqq g^2$). Here a number α in K may be said to be dominated by k provided there exists a number α_0 in k such that $|\alpha| < \alpha_0$. Choose $z_2, \cdots, z_\nu$ as numbers in the ring just mentioned!

As to 1), we observe that the matrix L defined by

$$L_\alpha = \eta_\alpha E \qquad (\eta_\alpha \text{ real number})$$

lies in $\mathfrak{L}_K$ according to the criterion, Lemma (3.5), if $\eta_\alpha = \eta_\beta$ holds for each pair of coördinated indices α, β (it lies even in $\mathfrak{L}$ when the η_α are the conjugates of a number η in the central field κ). The matrix L thus defined is *even*. Hence the assumption concerning $A_{\alpha\beta}$ implies the equation

$$(\eta_\alpha - \eta_\beta)A_{\alpha\beta} = 0.$$

Operating in $\mathfrak{L}_K$ one may choose $\eta_\alpha = 1$, $\eta_\beta = 0$ provided α and β are not coordinated; if one prefers to stay within $\mathfrak{L}$ one would take a determining number of $\kappa = k(\eta)$ for η and then have $\eta_\alpha \neq \eta_\beta$ under the same assumption. In either way one gets the desired result: $A_{\alpha\beta} = 0$.

Point 2) needs more careful consideration. We replace ϑ_α by the indeterminate root ϑ and for brevity's sake then suppress the argument ϑ (or the index α). With L ranging over all elements of $\mathfrak{L}_K(\vartheta)$, $C_0L = P$ varies over a linear set $\mathfrak{P}$. To the *even* L corresponds the *symmetric* P. The assumption that A commutes with L amounts to the relation

$$BP = PA \tag{6.21}$$

for the corresponding P when we put $C_0AC_0^{-1} = B$. Requiring (6.21) to hold for every symmetric P in $\mathfrak{P}$, makes superfluous the explicit statement of this link between the constant matrices A and B: $BC_0 = C_0A$, as it is included in (6.21) for $L = E$, $P = C_0$. However, we have to add the one equation

$$BP^0 = P^0A \tag{6.22}$$

corresponding to the fixed element $P^0 = C = C_0L_0$. Our concern is to ascertain that two matrices A, B satisfying (6.21) for P^0 and every *symmetric* P in $\mathfrak{P}$ satisfy (6.21) for every P in $\mathfrak{P}$.

Let q be our quantics forming the division algebra $\mathfrak{q}$ of order $d = 1, 2$ or 4, over K, and $Q = (q) \times E_f$. Each matrix $P = C_0L$, L in $\mathfrak{L}_K(\vartheta)$, satisfies the equation

$$Q'_*P = PQ; \tag{6.23}$$

and vice versa, a P satisfying (6.23) for each q must needs be $= C_0L$ where L commutes with each Q and hence belongs to the algebra $\mathfrak{L}_K(\vartheta)$. By the way, $\mathfrak{L}_K(\vartheta)$ is the algebra $(\mathfrak{q}')_f$ and each P may be written as NL where $N = (n) \times E_f$ is the constant "norm matrix" (4.32). In either way we find that the linear K-set $\mathfrak{P}$ consists of all matrices of the following form in the three cases $d = 1$, 2, 4, respectively:

$$P_0, \qquad \begin{Vmatrix} P_0, & -P_1 \\ P_1, & \lambda P_0 \end{Vmatrix}, \qquad \begin{Vmatrix} P_0, & -P_1, & -P_2, & -P_3 \\ P_1, & \lambda P_0, & P_3, & -\lambda P_2 \\ P_2, & -P_3, & \mu P_0, & \mu P_1 \\ P_3, & \lambda P_2, & -\mu P_1, & \lambda\mu P_0 \end{Vmatrix} \tag{6.24}$$

where P_0 or P_0, P_1 or P_0, P_1, P_2, P_3 are arbitrary real matrices of degree f. This is in agreement with the order $df^2 = f^2, 2f^2, 4f^2$. Such a P is *symmetric* provided P_0 is symmetric and P_1, P_2, P_3 are skew-symmetric. The question can now be settled by the following trivial

LEMMA (6.2). Two matrices A and B of degree f satisfying the equation $BX = XA$ for all symmetric X are of necessity the same multiple $A = B = \alpha E$ of the unit matrix, and hence satisfy the same equation for all X whatsoever.

PROOF: $X = E$ yields

$$B = A = \| a_{ik} \|.$$

With a diagonal matrix X of the elements $x_{ii} = x_i$ one gets

$$a_{ik} x_k = x_i a_{ik};$$

hence if $i \neq k$ by choosing $x_i = 1$, $x_k = 0$: $a_{ik} = 0$. Consequently A is a diagonal matrix of the elements $a_{ii} = a_i$. We finally obtain with an arbitrary symmetric $X = \| x_{ik} \|$:

$$(a_i - a_k) x_{ik} = 0,$$

therefore $a_i = a_k$.

In case I the lemma settles our question at once: the validity of (6.21) for all symmetric P's implies the same for all P's whatsoever. In case II and III we write

$$A = \| A_{ik} \|, \qquad (i, k = 0, 1 \text{ or } 0, 1, 2, 3)$$

the same for B, and

$$\lambda_0 = 1, \lambda_1 = \lambda \mid \lambda_0 = 1, \lambda_1 = \lambda, \lambda_2 = \mu, \lambda_3 = \lambda\mu.$$

We first take P_1 $(= P_2 = P_3) = 0$ and obtain the equations

$$B_{ik} \lambda_k P_0 = P_0 \lambda_i A_{ik} \tag{6.25}$$

holding for every symmetric P_0. Our lemma shows that therefore

$$A_{ik} = \alpha_{ik} E_f, \qquad B_{ik} = \beta_{ik} E_f$$

are multiples of the unit matrix, the real numbers α_{ik}, β_{ik} satisfying

$$\lambda_i \alpha_{ik} = \lambda_k \beta_{ik}.$$

Hence the equations (6.25) hold for every P_0 whatsoever. If we now consider the equation (6.21) for those P, (6.24), in which $P_0 = 0$ and if we treat P_1, P_2, P_3 as independent matrices, we find a certain number of equations

$$\alpha P_i = \beta P_i \qquad (i = 1 \text{ or } i = 1, 2, 3) \tag{6.26}$$

where α and β are numbers. They are required to hold good for an arbitrary anti-symmetric P_i. *If* $f > 1$ there exist anti-symmetric matrices $\neq 0$ and thus (6.26) implies $\alpha = \beta$; but then (6.26) holds for every P_i whatsoever, and we

thus made sure that validity of (6.21) for symmetric P's implies the same for all P's. *The case $f = 1$ is different.* Here we have only *one* independent symmetric P, and thus $\mathfrak{L}_K^+(\vartheta)$ consists of the multiples of the unit matrix alone, and so does its algebraic closure $(\mathfrak{L}_K^+(\vartheta))$. In case II, $f = 1$, one is forced to choose L_0 odd and the corresponding $P^0 = C = C_0L_0$ antisymmetric. Else (6.21) for all symmetric P's together with (6.22) would be bound to have more solutions than the equation (6.21) when required for *all* P's. In case III, $f = 1$, even this trick will not help us out of the trap. For even with an odd $L_0(\vartheta)$ the sum $(\mathfrak{L}_K^+(\vartheta)) + L_0^{-1}(\vartheta)(\mathfrak{L}_K^+(\vartheta))$ is of order 2 rather than of order 4, as it should be.

(6.3) The question whether there exists an odd *non-singular* $L_0(\vartheta)$ is to be discussed. In case I this is only possible for an *even* f. But for $d = 1, f$ even, or $d = 2$ or $d = 4$, (6.24) at once allows writing down a non-singular antisymmetric P and hence an odd $L_0(\vartheta)$ lying in the extension $\mathfrak{L}_K^-(\vartheta)$. The parts $L_\alpha = L_0(\vartheta_\alpha)$ and L_β corresponding to non-coördinated α and β may be chosen independently whereas for coördinated indices L_β is to be taken as $A_{\alpha\beta}^{-1}L_\alpha A_{\alpha\beta}$, or P_β as $e_{\alpha\beta}\cdot A'_{\alpha\beta}P_\alpha A_{\alpha\beta}$; one thus obtains an odd non-singular L_0 in $\mathfrak{L}_K$. If one expresses the unsplit L_0 in terms of a base of $\mathfrak{L}^-$ with certain real coefficients y, one sees that $|L_0|$ is not identically zero in the variables y. One therefore may ascertain rational values of the y for which $|L_0| \neq 0$; this L_0 is then an odd non-singular matrix of $\mathfrak{L}$. We summarize our construction in the

Main Theorem, Second Part. *When the Riemann algebra $\mathfrak{L}$ is described over a totally real splitting field $k(\vartheta)$ by means of a quantic factor set of the kind defined in the first part of the Main Theorem, then the norm of the factor set must be totally positive equivalent* 1. *This condition is not only necessary but also sufficient for $\mathfrak{L}$ to be associated with an even or odd pure Riemann matrix, save for the following limitations:*

$d = 1, f$ *odd*	$d = 2, f = 1$	$d = 4, f = 1$
no odd,	*no even,*	*neither an odd nor an even,*

associated Riemann matrix exists.

We must return for a moment to the investigation of necessary rather than sufficient conditions in order to determine whether these limitations lie in the nature of things and are not merely due to a lack of skill in our construction. To this end we have to consider that by Rosati's lemma (4.1), C necessarily decomposes into non-singular C_α's. The involution $q \to q_*$ effected by C_α in the realm of α-quantics is prescribed, hence $C(\vartheta)$ must be $= C_0(\vartheta)L(\vartheta)$ where $L(\vartheta)$ commutes with all $Q(\vartheta)$ and therefore must come to lie in $\mathfrak{L}(\vartheta)$ after $\mathfrak{L}(\vartheta)$ has been extended to $k(\vartheta)$. This leaves us no loophole.

7. Appendix. Automorphisms.

The scheme A as well as B, (1.34), may thus be described: it is a checkered square table with rows and columns labeled by a double index $i\alpha$ and $k\beta$ each

field of which is occupied by a d-rowed matrix $A_{i\alpha, k\beta}$. If E_{ik} denotes the unit or zero matrix according as $i = k$ or $i \neq k$ we have more precisely

$$(7.1) \qquad A_{i\alpha, k\beta} = A_{ik}E_{\alpha\beta}, \qquad B_{i\alpha, k\beta} = E_{ik}B_{\alpha\beta}.$$

Let us return for a moment to the irreducible representation $\mathfrak{A}$: $a \to A = A(a)$ of a simple algebra $\mathfrak{a}$. If an automorphism $a \to a^*$ of $\mathfrak{a}$ be given, then $a \to A(a^*) = A^*$ is a representation of $\mathfrak{a}$ as well as $\mathfrak{A}$: $a \to A(a)$ itself, and like any representation of the simple $\mathfrak{a}$ is equivalent to a multiple of $\mathfrak{A}$. The words "a multiple of" are to be canceled because of equality of degree g. Hence there exists a non-singular matrix H in k such that

$$(7.2) \qquad A^* = HAH^{-1}$$

for every A in $\mathfrak{A}$. This applies in particular

(α) to the full matric algebra $\mathfrak{M}_g$ consisting of all g-rowed matrices in k, and

(β) to the regular representation of a division algebra.

The same holds true for the *multiple* $s\mathfrak{A}$ of our irreducible $\mathfrak{A}$ which we now again call $\mathfrak{A} = \{A\}$: each automorphism $A \to A^*$ is of the type (7.2). The matrix H at the same time defines an automorphism $B \to B^*$ in the commutator algebra $\mathfrak{B}$: $B^* = HBH^{-1}$. So we are led to study *simultaneous* automorphisms $A \to A^*$, $B \to B^*$ in $\mathfrak{A}$ and $\mathfrak{B}$. A necessary condition that both are expressible in the form

$$(7.3) \qquad A^* = HAH^{-1}, \qquad B^* = HBH^{-1}$$

by the same non-singular constant H in k is their coincidence within the cross-cut $\mathfrak{Z}$ of $\mathfrak{A}$ and $\mathfrak{B}$, the so-called *centrum*. In formula (7.1) each A_{ik} varies over ($\mathfrak{d}'$), each $B_{\alpha\beta}$ over ($\mathfrak{d}$). An element A common to $\mathfrak{A}$ and $\mathfrak{B}$, must have $A_{ik} = J \cdot E_{ik}$ in (7.1) where J lies in ($\mathfrak{d}'$) and in ($\mathfrak{d}$):

$$J: x \to x' = j_1 x = x j_2$$

(j_1 and j_2 fixed elements, x variable in $\mathfrak{d}$). But $j_1 x = x j_2$ yields $j_1 = j_2$ by putting $x = e$, and $j = j_1 = j_2$ must commute with all elements x of $\mathfrak{d}$. The elements j of this kind form the *centrum* $\mathfrak{z}$ of $\mathfrak{d}$. *Let us first assume that* $\mathfrak{z}$ *is of order* 1, that only the numerical multiples of the unit element e commute with all elements x of the division algebra $\mathfrak{d}$.

We then maintain that the d^2 transformations

$$(7.4) \qquad x' = bxa$$

yield a base of the complete matric algebra $\mathfrak{M}_d$ if we let a and b run independently over a base of $\mathfrak{d}$. By Burnside's theorem this is true provided the multiples of the unit matrix are the only transformations J commuting with all these transformations (7.4), i.e. with all transformations of type $x' = bx$ and $x' = xa$. For the first reason such a J must be itself of the form $x' = xj_2$, for the second reason of the form $x' = j_1 x$ (j_1 and j_2 in $\mathfrak{d}$); hence $j_1 = j_2$ lies in the centrum of $\mathfrak{d}$ and is a multiple of e. The result is that the product $A_{11}B_{11}$ yields

a full base for all d-rowed matrices when A_{11} ranges over a base for $(\mathfrak{d}')$ and B_{11} for $(\mathfrak{d})$. The product of two matrices A and B, (7.1), is given by

$$(AB)_{i\alpha, k\beta} = A_{ik} \cdot B_{\alpha\beta},$$

and from this formula in connection with the result just obtained we readily deduce that AB provides a full base for all g-rowed matrices ($g = dst$) if A runs over a base of $\mathfrak{A}$ and B of $\mathfrak{B}$. This is in keeping with the orders $d \cdot t^2$ of $\mathfrak{A}$ and $d \cdot s^2$ of $\mathfrak{B}$; for their product equals $(dst)^2 = g^2$.

The two arbitrary given automorphisms $A \to A^*$, $B \to B^*$ define therefore (remembering that the A's commute with the B's!) an automorphism $AB \to A^*B^*$ of the full matric algebra $\mathfrak{M}_g$ and consequently statement (α) above assures us of the existence of a constant non-singular matrix H such that $A^*B^* = HABH^{-1}$, in particular (A or $B = E$):

$$A^* = HAH^{-1}, \qquad B^* = HBH^{-1}.$$

H is unambiguously determined, but for a numerical factor.

When we combine the identical automorphism of $\mathfrak{B}$ with a given automorphism $A \to A^*$ of $\mathfrak{A}$, our H commutes with every B and hence lies in $\mathfrak{A}$: *Every automorphism of $\mathfrak{A}$ is an inner automorphism.*[10]

If the centrum $\mathfrak{z}$ of $\mathfrak{d}$ is of order δ we may consider $\mathfrak{d}$ as a division algebra of order d/δ over the *field* $\mathfrak{z}$. Operating in this field throughout and finally replacing again each "number" j of this field by the δ-rowed matrix that represents it in the regular representation of $\mathfrak{z}$, we carry over our result to each pair of automorphisms $A \to A^*$, $B \to B^*$ in $\mathfrak{A}$ and $\mathfrak{B}$ which coincide with the identity for the elements Z common to $\mathfrak{A}$ and $\mathfrak{B}$. Application of the statement (β) above to the commutative division algebra $\mathfrak{z}$ enables us to weaken this restricting hypothesis to the assumption that both automorphisms coincide among each other for the elements Z of $\mathfrak{Z}$:

THEOREM. *Two automorphisms $A \to A^*$, $B \to B^*$ of $\mathfrak{A}$ and $\mathfrak{B}$ when coinciding within the centrum or cross-cut $\mathfrak{Z}$ of $\mathfrak{A}$ and $\mathfrak{B}$ are generated by the same non-singular matrix H according to* (7.3). *In particular, each automorphism of $\mathfrak{A}$ which leaves invariant the elements of $\mathfrak{Z}$ is an inner automorphism.*

It is in no way unnatural that the proof first deals with the case of a "normal" algebra whose centrum does not reach beyond the reference field k. For what ambiguity there is in H comes from the centrum: the unruly things happen in the commutative fields, the whole superstructure of algebras is of a comparatively simple nature.

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[10] Skolem, "Zur Theorie der assoziativen Zahlensysteme," *Skr. Norske Vid.-Akad.*, Oslo (1927), pp. 21, 22; R. Brauer, *Math. Zeitschrift*, vol. **30** (1929), p. 105.